Uniform sup-norm bounds on average for Siegel cusp forms

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Abstract

Let $\Gamma \subsetneq \mathrm{Sp}_n(\mathbb{R})$ be an arithmetic subgroup of the symplectic group $\mathrm{Sp}_n(\mathbb{R})$ acting on the Siegel upper half-space $\mathbb{H}_n$ of degree n. Consider the d-dimensional space of Siegel cusp forms $\mathcal{S}_\kappa^n(\Gamma)$ of weight κ for Γ and let $\{f_j\}_{1 \leq j \leq d}$ be a basis of $\mathcal{S}_\kappa^n(\Gamma)$ orthonormal with respect to the Petersson inner product. In this paper we show using the heat kernel method that the sup-norm of the quantity $S_\kappa^\Gamma(Z) := \sum_{j=1}^d \det(Y)^\kappa |f_j(Z)|^2$ $(Z \in \mathbb{H}_n)$ is bounded above by $c_{n,\Gamma} \kappa^{n(n+1)/2}$ when $M := \Gamma \backslash \mathbb{H}_n$ is compact and by $c_{n,\Gamma} \kappa^{3n(n+1)/4}$ when M is non-compact of finite volume, where $c_{n,\Gamma}$ denotes a positive real constant depending only on the degree n and the group Γ. Furthermore, we show that this bound is uniform in the sense that if we fix a group Γ_0 and take Γ to be a subgroup of Γ_0 of finite index, then the constant $c_{n,\Gamma}$ in these bounds depends only on the degree n and the fixed group Γ_0.

1 Introduction

Obtaining sup-norm bounds $\|\varphi\|_\infty$ of eigenfunctions φ satisfying $\Delta_X \varphi + \lambda \phi = 0$ for the Laplace–Beltrami operator Δ_X on a Riemannian manifold X in terms of the eigenvalue λ is a classical problem in spectral theory, for which local estimates exist [21, 34] that are essentially sharp in this level of generality. In arithmetic setting, these estimates are expected to be improved drastically[33, 32]. Although major improvements over the classical general estimates have been obtained in such setting [23], we are still far from the conjectured bound

$$\|\varphi\|_\infty \ll_\epsilon \lambda^\epsilon \qquad (\epsilon > 0) \tag{1.1}$$

for Hecke eigenforms. However, more interestingly, in this arithmetic setting, the sup-norm bound problem has been shown to have important connections to various fundamental queries in number theory such as the Lindelöf hypothesis for the Riemann zeta function [32], quantum ergodicity and entropy bounds [6], the subconvexity problem for L-functions [17], distribution of zeros of modular forms [16] and the study of Arakelov invariants of arithmetic surfaces [1, 24, 29]. This has created a sustained interest in the sup-norm bound problem in various number-theoretic aspects, one of which we address in this paper.

Although we are far from obtaining (1.1) for individual eigenforms, in the special setting of holomorphic cusp forms, in [11], optimal bounds have been obtained on average over an orthonormal basis, without the assumption of strong arithmetic symmetries such as Hecke structure on the eigenforms. As this setup has a ready generalization in the case of Siegel modular forms, where the sup-norm bound has also recently been of some interest[4, 5, 7, 8], we attempt here to extend this method and the results obtained in [11] to the case of the Siegel upper half-space, which works to a large extent along with some significant non-trivialities that need special tools from the theory of harmonic analysis of semisimple Lie groups to get around.

1.1 Sup-norm bounds on $\mathbb{H}$

Let $\mathbb{H} := \{z = x + iy \mid y > 0\}$ denote the upper half-plane and $\Gamma \subsetneq \mathrm{SL}_2(\mathbb{R})$ denote a Fuchsian subgroup of the first kind. Let $\mathcal{V}_\kappa(\Gamma)$ denote the space of real analytic functions $\varphi \colon \mathbb{H} \to \mathbb{C}$ with the transformation behaviour

$$\varphi(\gamma z) = \left(\frac{cz+d}{c\bar{z}+d}\right)^{\kappa/2} \varphi(z) \qquad \left(\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma\right). \tag{1.2}$$

For $\varphi \in \mathcal{V}_\kappa(\Gamma)$, we set

$$\|\varphi\|^2 := \int\limits_{\Gamma \backslash \mathbb{H}} |\varphi(z)|^2 \, \frac{\mathrm{d}x \wedge \mathrm{d}y}{y^2},$$

whenever it is defined. Let $\mathcal{H}_\kappa(\Gamma) := \{\varphi \in \mathcal{V}_\kappa(\Gamma) \mid \|\varphi\| < \infty\}$ denote the Hilbert space of square integrable functions in $\mathcal{V}_\kappa(\Gamma)$, equipped with the Petersson inner product. Let $\Delta^{(\kappa)}$ denote the Maaß Laplacian invariant with respect to the action of Γ on φ in (1.2). Then the operator $\Delta^{(\kappa)}$ acts on the smooth functions of $\mathcal{H}_\kappa(\Gamma)$ and extends to an essentially self-adjoint linear operator acting on a dense subspace of $\mathcal{H}_\kappa(\Gamma)$.

The eigenvalues for the Laplace equation $(\Delta^{(\kappa)} + \lambda)\varphi = 0$ satisfy $\lambda \geq \kappa/2(1 - \kappa/2)$ and in case $\lambda = \kappa/2(1-\kappa/2)$, the corresponding eigenfunction φ in $\mathcal{H}_\kappa(\Gamma)$ can be shown to be of the form $\varphi(z) = y^{\kappa/2} f(z)$ with $f \in \mathcal{S}_\kappa(\Gamma)$, where $\mathcal{S}_\kappa(\Gamma)$ denotes the space of holomorphic cusp forms of weight κ with respect to Γ.

Let $d = \dim(\mathcal{S}_\kappa(\Gamma))$ be the dimension of the space $\mathcal{S}_\kappa(\Gamma)$ and consider a basis $\{f_j\}_{1 \leq j \leq d}$ of $\mathcal{S}_\kappa(\Gamma)$ orthonormal with respect to the Petersson inner product. Then, in order to obtain a sup-norm bound for the quantity

$$S_\kappa^\Gamma(z) := \sum_{j=1}^d y^\kappa |f_j(z)|^2 \quad (z \in \mathbb{H})$$

in the weight aspect, using the spectral decomposition of the weight-κ heat kernel $K_t^{(\kappa,\Gamma)}$ and the $\mathbb{C}$-vector space isomorphism

$$\mathcal{S}_\kappa(\Gamma) \cong \ker\left(\Delta^{(\kappa)} + \frac{\kappa}{2}\left(1 - \frac{\kappa}{2}\right)\mathrm{id}\right) \tag{1.3}$$

induced by the assignment $f \mapsto y^{\kappa/2}f$, one arrives at the important relation

$$S_\kappa^\Gamma(z) = \lim_{t \to \infty} \exp\left(-\frac{\kappa}{2}\left(\frac{\kappa}{2} - 1\right)t\right) K_t^{(\kappa,\Gamma)}(z,z) \qquad (\kappa \geq 2), \tag{1.4}$$

whence analyzing the heat kernel $K_t^{(\kappa,\Gamma)}$, in [11] it is shown that

$$\sup_{z \in \mathbb{H}} S_\kappa^\Gamma(z) \leq \begin{cases} c_\Gamma \, \kappa & (M \text{ compact}), \\ c_\Gamma \, \kappa^{3/2} & (M \text{ non-compact of finite volume}), \end{cases}$$

where $c_\Gamma > 0$ is a positive real number depending only on Γ. Furthermore, it is shown that this bound is uniform in the sense that if we fix a group $\Gamma_0 \subsetneq \mathrm{SL}_2(\mathbb{R})$ and take Γ to be a subgroup of Γ_0 of finite index, then the constant c_Γ in these bounds depends only on the fixed group Γ_0.

1.2 Sup-norm bounds on $\mathbb{H}_n$

Let $\mathbb{H}_n := \{Z = X + iY \in \mathbb{C}^{n \times n} \mid X, Y \in \mathrm{Sym}_n(\mathbb{R}) : Y > 0\}$ denote the Siegel upper half-space of degree n and $\Gamma \subsetneq \mathrm{Sp}_n(\mathbb{R})$ be an arithmetic subgroup of the symplectic group $\mathrm{Sp}_n(\mathbb{R})$. Let $\mathcal{S}_\kappa^n(\Gamma)$ denote the space of all Siegel cusp forms of weight κ. Let $d = \dim(\mathcal{S}_\kappa^n(\Gamma))$ be the dimension of the space $\mathcal{S}_\kappa^n(\Gamma)$ and $\{f_j\}_{1 \leq j \leq d}$ be an orthonormal basis of $\mathcal{S}_\kappa^n(\Gamma)$ with respect to the Petersson inner product. We denote

$$S_\kappa^\Gamma(Z) := \sum_{j=1}^d \det(Y)^\kappa |f_j(Z)|^2 \quad (Z \in \mathbb{H}_n).$$

In [7], for $\Gamma = \Gamma_n = \mathrm{Sp}_n(\mathbb{Z})$, an asymptotic analysis of the Bergman kernel shows the bound

$$\sup_{Z \in K} S_\kappa^\Gamma(Z) \asymp_n \kappa^{n(n+1)/2} \qquad (\kappa > 2n) \tag{1.5}$$

in the weight aspect, where $K \subsetneq \mathscr{F}_n$ is any fixed compact subset of the standard fundamental domain $\mathscr{F}_n$ of $\mathbb{H}_n$ for Γ_n.

In case of individual forms, on the basis of upper and lower bounds for certain specific kinds of Siegel cusp forms $F \in \mathcal{S}_\kappa^n(\Gamma)$, namely those coming from elliptic modular forms $f \in \mathcal{S}_\kappa(\Gamma)$ via Ikeda lifts, in the weight aspect, for L^2-normalized Siegel Hecke cusp forms F for Γ_n of (large) weight κ and (fixed) genus n, Blomer in [4] conjectures

$$\sup_{Z \in \mathbb{H}_n} \det(Y)^\kappa |F(Z)|^2 \asymp_n \kappa^{n(n+1)/4}.$$

Combining this conjecture with Hashimoto's result [19]

$$\dim_\mathbb{C} \mathcal{S}_\kappa^n(\Gamma) = 2^{n(n-1)/2} \frac{\mathrm{vol}\left(\Gamma\backslash\mathbb{H}_n\right)}{(4\pi)^{n(n+1)/2}} \kappa^{n(n+1)/2} + O(\kappa^{n(n+1)/2-1}),$$

for $S_\kappa^\Gamma(Z)$ one conjectures

$$\sup_{Z \in \mathbb{H}_n} S_\kappa^\Gamma(Z) = O_\Gamma(\kappa^{3n(n+1)/4}). \tag{1.6}$$

Recently, in [8] it has been shown that

$$\left.\begin{array}{c} \kappa^{3n(n+1)/4} \\ \kappa^{3n(n+1)/4} \\ \kappa^{3n(n+1)/4} \end{array}\right\} \ll_n \sup_{Z \in \mathbb{H}_n} S_\kappa^{\Gamma_n}(Z) \ll_{n,\epsilon} \left\{\begin{array}{ll} \kappa^{3n(n+1)/4} & (n=1) \\ \kappa^{3n(n+1)/4+\epsilon} & (n=2) \\ \kappa^{(5n-3)(n+1)/4+\epsilon} & (n \geq 3) \end{array}\right.,$$

which establishes (1.6) for $n=1$ and $n=2$, but moves away from the optimal upper bound for $n > 2$.

1.3   Statement of results

The main result of this paper is the following theorem, which establishes the conjecture (1.6) for $n \geq 2$ by relating $S_\kappa^\Gamma(Z)$ with the heat kernel $K_t^{(\kappa,\Gamma)}$ corresponding to the Siegel–Maaß Laplacian $\Delta^{(\kappa)}$ on $\Gamma\backslash\mathbb{H}_n$.

Theorem 1.1. *Let $\Gamma \subsetneq \mathrm{Sp}_n(\mathbb{R})$ be an arithmetic subgroup and $\mathcal{S}_\kappa^n(\Gamma)$ denote the space of Siegel cusp forms of weight κ on $\mathbb{H}_n$ with respect to Γ. Let $\{f_j\}_{1 \leq j \leq d}$ be a basis of $\mathcal{S}_\kappa^n(\Gamma)$ orthonormal with respect to the Petersson inner product. Then, for all $n \geq 2$, we have*

$$\sup_{Z \in \mathbb{H}_n} \sum_{j=1}^d \det(Y)^\kappa |f_j(Z)|^2 \leq \begin{cases} c_{n,\Gamma}\, \kappa^{n(n+1)/2} & (\Gamma \text{ cocompact}), \\ c_{n,\Gamma}\, \kappa^{3n(n+1)/4} & (\Gamma \text{ cofinite}), \end{cases} \quad (\kappa \geq n+1)$$

where $c_{n,\Gamma} > 0$ is a positive real number depending only on the degree n of $\mathbb{H}_n$ and the group Γ.

Furthermore, this bound is uniform in the sense that if we fix a group $\Gamma_0 \subsetneq \mathrm{Sp}_n(\mathbb{R})$ and take Γ to be a subgroup of Γ_0 of finite index, then the constant $c_{n,\Gamma}$ in these bounds depends only on the degree n and the fixed group Γ_0.

This generalizes the theorems 4.2, 5.2, and 6.1 in [11]. Furthermore, as $\kappa \geq n+1$, essentially the same arguments generalize the theorem 3.1 in [25] to obtain for an orthonormal basis $\{f_j\}_{1 \leq j \leq d}$ of $\mathcal{S}_{n+1}^n(\Gamma)$ and a positive real number $c_{n,\Gamma_0} > 0$ depending only on n and a fixed base space $M_0 := \Gamma_0\backslash\mathbb{H}_n$, the estimate

$$\frac{\mathrm{d}\mu_B(Z)}{\mathrm{d}\mu_S(Z)} = \sup_{Z \in \mathbb{H}_n} \sum_{j=1}^d \det(Y)^{n+1} |f_j(Z)|^2 \leq c_{n,\Gamma_0},$$

where $\mathrm{d}\mu_B$ denotes the volume form of the Bergman metric

$$\mathrm{d}\mu_B(Z) := \sum_{j=1}^d |f_j(Z)|^2 \bigwedge_{1 \leq j \leq k \leq n} \mathrm{d}x_{j,k} \wedge \mathrm{d}y_{j,k}$$

and $\mathrm{d}\mu_S$ denotes the volume form of the Siegel metric

$$\mathrm{d}\mu_S(Z) := \frac{\displaystyle\bigwedge_{1 \leq j \leq k \leq n} \mathrm{d}x_{j,k} \wedge \mathrm{d}y_{j,k}}{\det(Y)^{n+1}}$$

on $\mathbb{H}_n$.

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1.4 Strategy of the proof

For the proofs, we follow the same general method developed in [11]. Let $\mathcal{V}_\kappa^n(\Gamma)$ denote the space of all real-analytic functions $\varphi\colon \mathbb{H}_n \to \mathbb{C}$, which have the transformation behaviour

$$\varphi(\gamma Z) = \left(\frac{\det(CZ + D)}{\det(C\overline{Z} + D)} \right)^{\kappa/2} \varphi(Z) \qquad \left(\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma \right). \tag{1.7}$$

For $\varphi \in \mathcal{V}_\kappa^n(\Gamma)$, we set

$$\|\varphi\|^2 := \int\limits_{\Gamma \backslash \mathbb{H}_n} |\varphi(Z)|^2 \, \mathrm{d}\mu_n(Z),$$

whenever it is defined. Let $\mathcal{H}_\kappa^n(\Gamma) := \{ \varphi \in \mathcal{V}_\kappa^n(\Gamma) \mid \|\varphi\| < \infty \}$ denote the Hilbert space of square integrable functions in $\mathcal{V}_\kappa^n(\Gamma)$, equipped with the Petersson inner product. Let $\Delta^{(\kappa)}$ denote the Siegel-Maaß Laplacian invariant with respect to the action of Γ on φ in (1.7). Then, the operator $\Delta^{(\kappa)}$ acts on the smooth functions of $\mathcal{H}_\kappa^n(\Gamma)$ and extends to an essentially self-adjoint linear operator acting on a dense subspace of $\mathcal{H}_\kappa^n(\Gamma)$.

The eigenvalues for the Laplace equation $(\Delta^{(\kappa)} + \lambda)\,\varphi = 0$ satisfy the inequality $\lambda \geq (n\kappa/4)\big((n+1) - \kappa\big)$, with the equality $\lambda = (n\kappa/4)\big((n+1) - \kappa\big)$ being attained if and only if φ is of the form $\varphi(Z) = \det(Y)^{\kappa/2} f(Z)$ for some Siegel cusp form $f \in \mathcal{S}_\kappa^n(\Gamma)$ of weight κ, i.e., the $\mathbb{C}$-vector space isomorphism

$$\mathcal{S}_\kappa^n(\Gamma) \cong \ker\left(\Delta^{(\kappa)} + \frac{n\kappa}{4}((n+1) - \kappa)\mathrm{id} \right) \tag{1.8}$$

induced by the assignment $f \mapsto \det(Y)^{\kappa/2} f$ holds (See [27, corollary 5.4]).

Then, in a manner similar to (4.1), we use the spectral decomposition of the heat kernel $K_t^{(\kappa,\Gamma)}$ corresponding to the Siegel–Maaß Laplacian $\Delta^{(\kappa)}$ on $\Gamma \backslash \mathbb{H}_n$ to generalize the relation in (1.4) to obtain

$$S_\kappa^\Gamma(Z) = \lim_{t \to \infty} \exp\left(-\frac{n\kappa}{4}(\kappa - (n+1))\, t \right) K_t^{(\kappa,\Gamma)}(Z, Z). \tag{1.9}$$

As, for $t > 0$, both the function $\exp(-n\kappa(\kappa - (n+1))\, t/4)$ and the heat kernel $K_t^{(\kappa,\Gamma)}$ are monotonically decreasing in t, from (1.9), one also obtains the inequality

$$S_\kappa^\Gamma(Z) \leq \exp\left(-\frac{n\kappa}{4}(\kappa - (n+1))\, t \right) K_t^{(\kappa,\Gamma)}(Z) \qquad (t > 0), \tag{1.10}$$

whence analyzing the heat kernel $K_t^{(\kappa,\Gamma)}$, we arrive at the results stated in Theorem 1.1.

The non-triviality in extending these results from $n = 1$ to $n \geq 1$ lies in constructing the heat kernel $K_t^{(\kappa)}$ corresponding to the Siegel–Maaß Laplacian $\Delta^{(\kappa)}$ on $\mathbb{H}_n$ from which the heat kernel $K_t^{(\kappa,\Gamma)}$ on $\Gamma \backslash \mathbb{H}_n$ is obtained by periodization. We use a method of calculating spherical functions on real semisimple Lie groups by reducing them to the complex case developed by Flensted-Jensen in [9] to construct a spherical function for $\Delta^{(\kappa)}$ on $\mathbb{H}_n$ and then use the traditional method of obtaining the heat kernel from a spherical function to construct $K_t^{(\kappa)}$. The spherical function and the ensuing heat kernel so constructed are not totally explicit, as they involve a change of variable that is somewhat implicit in nature. In the end, we got around this difficulty in the cocompact case by using a counting function argument to estimate the periodization sum in $K_t^{(\kappa,\Gamma)}$ with an integral of $K_t^{(\kappa)}$ over the radial coordinates, which allows us to change back from the implicit change of variables. In the cofinite case, we consider a limiting case of $K_t^{(\kappa)}$, which can be constructed explicitly. Thankfully, these special cases seem to suffice for our purpose.

1.5 Brief outline of the paper

In section 2 we gather some basic preliminaries on the symplectic group, Siegel upper half-space and Siegel modular forms for later use in our calculations. In section 3 we gather the background on spherical functions and construction of the heat kernel on symmetric spaces. Then we use the Flensted-Jensen reduction method of calculating spherical functions on real semisimple groups via complex semisimple

groups to construct the spherical function as well as the heat kernel on the Siegel upper half-space. All these calculations being for the Laplace–Beltrami operator Δ, next we correct them for weight-κ to obtain the heat kernel on Siegel upper half-space corresponding to the Siegel–Maass Laplacian $\Delta^{(\kappa)}$. Finally, in section 4, we analyze this weight-κ heat kernel to obtain uniform sup-norm bounds for Siegel cusp forms on average over an orthonormal basis in both cocompact and cofinite cases.

1.6 Acknowledgement

2 Background on Siegel modular forms

In this section, we gather some basic preliminaries on the symplectic group, Siegel upper half-space and Siegel modular forms from some standard references such as [10], [26] and [35].

2.1 Siegel upper half-space

For $n \in \mathbb{N}_{>0}$ and a commutative ring R, let $R^{m \times n}$ denote the set of $(m \times n)$-matrices with entries in R and $\mathrm{Sym}_n(R)$ denote the set of symmetric matrices in $R^{n \times n}$. The Siegel upper half-space $\mathbb{H}_n$ of degree n is then defined by

$$\mathbb{H}_n := \{Z = X + iY \in \mathbb{C}^{n \times n} \,|\, X, Y \in \mathrm{Sym}_n(\mathbb{R}) : Y > 0\}.$$

The symplectic group $\mathrm{Sp}_n(\mathbb{R})$ of degree n is defined by

$$\mathrm{Sp}_n(\mathbb{R}) := \{g \in \mathbb{R}^{2n \times 2n} \,|\, g^t J_n g = J_n\},$$

where $J_n \in \mathbb{R}^{2n \times 2n}$ is the skew-symmetric matrix

$$J_n := \begin{pmatrix} 0 & \mathbb{1}_n \\ -\mathbb{1}_n & 0 \end{pmatrix}$$

with $\mathbb{1}_n$ denoting the identity matrix of $\mathbb{R}^{n \times n}$. The group $\mathrm{Sp}_n(\mathbb{R})$ acts by the symplectic action

$$\mathbb{H}_n \ni Z \mapsto gZ = (AZ + B)(CZ + D)^{-1} \qquad \left(g = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{Sp}_n(\mathbb{R})\right) \tag{2.1}$$

on $\mathbb{H}_n$. Under this action $\mathrm{Im}(Z)$ transforms as

$$\mathrm{Im}(gZ) = (CZ + D)^{-t} \mathrm{Im}(Z)(C\overline{Z} + D)^{-1}, \tag{2.2}$$

which, on taking determinants on both sides, gives

$$\det(\mathrm{Im}(gZ)) = \frac{\det(\mathrm{Im}(Z))}{|\det(CZ + D)|^2}.$$

Similarly, taking matrix-differentials on both sides of the symplectic action (2.1), it is easy to see that under this action, the matrix-differential form $\mathrm{d}Z$ transforms as

$$\mathrm{d}(gZ) = (CZ + D)^{-t} \, \mathrm{d}Z \, (CZ + D)^{-1}. \tag{2.3}$$

The arclength $\mathrm{d}s_n^2$ and the volume form $\mathrm{d}\mu_n$ on $\mathbb{H}_n$ in terms of $Z = (z_{j,k})_{1 \le j \le k \le n} \in \mathbb{H}_n$ are given by

$$\mathrm{d}s_n^2(Z) = \mathrm{tr}(Y^{-1}\,\mathrm{d}Z\,Y^{-1}\,\mathrm{d}\overline{Z}) \qquad (Z = X + iY),$$

$$\mathrm{d}\mu_n(Z) = \frac{\bigwedge\limits_{1 \le j \le k \le n} \mathrm{d}x_{j,k} \wedge \mathrm{d}y_{j,k}}{\det(Y)^{n+1}} \qquad (z_{j,k} = x_{j,k} + iy_{j,k}). \tag{2.4}$$

From equations (2.2) and (2.3) it is obvious that the arclength $\mathrm{d}s_n^2$ and the volume form $\mathrm{d}\mu_n$ on $\mathbb{H}_n$ given by the above equations are invariant under the symplectic action. Corresponding to this metric, we have the Laplace–Beltrami operator

$$\Delta = \mathrm{tr}\left(Y\left(\left(Y\,\frac{\partial}{\partial X}\right)^t \frac{\partial}{\partial X} + \left(Y\,\frac{\partial}{\partial Y}\right)^t \frac{\partial}{\partial Y}\right)\right)$$

on $\mathbb{H}_n$, called the Siegel Laplacian, which is also invariant under the symplectic action.

The geodesic distance $s(Z, W)$ between the points $Z, W \in \mathbb{H}_n$ is given by

$$s(Z, W) = \sqrt{2}\left(\sum_{j=1}^n \log^2\left(\frac{1 + \sqrt{\rho_j}}{1 - \sqrt{\rho_j}}\right)\right)^{1/2},$$

where $\rho_j\,(1 \le j \le n)$ are the eigenvalues of the cross-ratio matrix

$$\rho(Z, W) = (Z - W)(\overline{Z} - W)^{-1}(\overline{Z} - \overline{W})(Z - \overline{W})^{-1} \quad (Z, W \in \mathbb{H}_n). \tag{2.5}$$

Remark 2.1. Due to the action (2.1) of $\mathrm{Sp}_n(\mathbb{R})$ on $\mathbb{H}_n$, the Siegel upper half-space can be viewed as a Riemannian globally symmetric space G_0/K_0, where $G_0 = \mathrm{Sp}_n(\mathbb{R})$ and $K_0 = \mathrm{Sp}_n(\mathbb{R}) \cap O_{2n}(\mathbb{R}) \cong U_n$ is the maximal compact subgroup of $\mathrm{Sp}_n(\mathbb{R})$ fixing the origin $i\mathbb{1}_n \in \mathbb{H}_n$. An explicit correspondence between G_0/K_0 and $\mathbb{H}_n$ is given by $gK_0 \mapsto g\,i\mathbb{1}_n$. Under this identification, the action of G_0 by left translation on G_0/K_0 translates to the symplectic action (2.1) of G_0 on $\mathbb{H}_n$. The space G_0/K_0 is equipped with a natural metric coming from the Killing form on the Lie algebra $\mathfrak{g}_0 = \mathfrak{sp}_n(\mathbb{R})$ of G_0. Under this identification, this metric takes the form of the metric given by (2.4) on $\mathbb{H}_n$.

2.2 Arithmetic subgroups and fundamental domains

A subgroup Γ of $\mathrm{Sp}_n(\mathbb{R})$ is called *discrete* if it acts discontinuously on $\mathbb{H}_n$, i.e., the orbit $\Gamma Z = \{\gamma Z \mid \gamma \in \Gamma\}$ has no accumulation point in $\mathbb{H}_n$, or equivalently, for any two compact sets $K_1, K_2 \subset \mathbb{H}_n$, the set $\{\gamma \in \Gamma \mid \gamma(K_1) \cap K_2 \ne \emptyset\}$ is finite. The most important example of a discrete subgroup of $\mathrm{Sp}_n(\mathbb{R})$ is the Siegel modular group $\Gamma_n := \mathrm{Sp}_n(\mathbb{Z})$.

Definition 2.2. A subgroup $\Gamma \subsetneq \mathrm{Sp}_n(\mathbb{R})$ is called an arithmetic subgroup if Γ is commensurable to Γ_n, i.e., the group $\Gamma \cap \Gamma_n$ has finite index in both Γ and Γ_n.

Because of their commensurability with the discrete subgroup Γ_n, arithmetic subgroups of $\mathrm{Sp}_n(\mathbb{R})$ are also discrete.

Any arithmetic subgroup Γ of $\mathrm{Sp}_n(\mathbb{R})$ has a fundamental domain, but it is not unique. A fundamental domain of the Siegel modular group Γ_n can be explicitly constructed by means of reduction theory applied to the positive definite imaginary part Y of $Z \in \mathbb{H}_n$. A vector $h^t = (h_1, h_2, \ldots, h_g) \in \mathbb{Z}^n$ is called primitive if for $1 \le k \le n$, we have $\gcd(h_k, \ldots, h_n) = 1$.

Definition 2.3. A positive definite matrix $Y = (y_{j,k})_{1 \le j,k \le n} \in \mathrm{P}_n$ is called Minkowski reduced if it satisfies $y_{k,k+1} \ge 0\,(1 \le k \le n-1)$ and for all primitive vectors $h \in \mathbb{Z}^n$, we have $h^t Y h \ge y_{k,k}\,(1 \le k \le n)$.

Proposition 2.4. *A Minkowski reduced positive definite matrix Y satisfies the properties*

(i) $y_{1,1} \le y_{2,2} \le \ldots \le y_{n,n}$,

(ii) $\left|2y_{j,k}\right| \le y_{j,j} \quad (1 \le j < k \le n)$,

(iii) $y_{k,k+1} \ge 0 \quad (1 \le k \le n-1)$,

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(iv) there exists a positive number $c_1(n)$ depending only on n such that

$$\det(Y) \leq \prod_{j=1}^{n} y_{j,j} \leq c_1(n) \det(Y),$$

(v) there exists a positive number $c_2(n)$ depending only on n such that

$$c_2(n)^{-1} Y < Y^{\mathrm{D}} < c_2(n) Y,$$

where Y^{D} denotes the diagonal matrix made up of the diagonal elements $y_{1,1}, \ldots, y_{n,n}$ of the matrix Y, i.e.,

$$Y^{\mathrm{D}} = \begin{pmatrix} y_{1,1} & & 0 \\ & \ddots & \\ 0 & & y_{n,n} \end{pmatrix}.$$

Proof. See [10, Satz 2.5, Folgerung 2.6], [26, page 20]. $\qquad\square$

Proposition 2.5. *The set of points $Z = X + iY \in \mathbb{H}_n$ satisfying the following criteria forms a fundamental domain $\mathscr{F}_n$ of the Siegel modular group Γ_n:*

(i) $|\det(CZ + D)| \geq 1$ for all $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma_n$,

(ii) $Y = \mathrm{Im}(Z)$ is Minkowski reduced,

(iii) for all $1 \leq j, k \leq n$, the matrix $X = (x_{j,k})_{1 \leq j,k \leq n}$ satisfies $|x_{j,k}| \leq 1/2$.

Proof. See [10, Satz 2.9]. $\qquad\square$

The fundamental domain $\mathscr{F}_n$ of Γ_n is called the standard fundamental domain of Γ_n and the matrices $Z \in \mathscr{F}_n$ are called Siegel reduced.

Proposition 2.6. *If $Z = X + iY \in \mathbb{H}_n$ is also Siegel reduced, then Y satisfies the properties*

(i) $y_{1,1} \geq \sqrt{3}/2$.

(ii) there exists a constant $c_3(n) > 0$ depending only on n, such that $Y \geq c_3(n) \mathbb{1}_n$.

Proof. See [10, Hilfssatzs 2.11 ,2.12]. $\qquad\square$

Using the fundamental domain $\mathscr{F}_n$ of the Siegel modular group Γ_n, we can construct fundamental regions of other arithmetic subgroups of Γ of $\mathrm{Sp}_n(\mathbb{R})$. Consider the space of the left cosets $\{\Gamma\gamma \mid \gamma \in \Gamma_n\}$. Since $\Gamma\gamma_1 = \Gamma\gamma_2$ if and only if $(\Gamma \cap \Gamma_n)\gamma_1 = (\Gamma \cap \Gamma_n)\gamma_2$ and $[\Gamma_n : \Gamma \cap \Gamma_n] < \infty$, they have a finite system of representatives $\gamma_1, \gamma_2, \ldots, \gamma_m$ $(m \in \mathbb{N}_{\geq 1})$. Then,

$$\mathscr{F} = \bigcup_{j=1}^{m} \gamma_j \mathscr{F}_n \tag{2.6}$$

is a fundamental region of Γ.

2.3 Boundary of the Siegel upper half space

The Siegel upper half-space $\mathbb{H}_n$ can be realized as bounded domain $\mathbb{D}_n = \{\zeta \in \mathrm{Sym}_n(\mathbb{C}) \mid \zeta\bar{\zeta} < \mathbb{1}_n\}$ through the Cayley transformation $l \colon \mathbb{H}_n \xrightarrow{\sim} \mathbb{D}_n$ given by the assignment

$$Z \mapsto \zeta = (Z - i\mathbb{1}_n)(Z + i\mathbb{1}_n)^{-1}, \tag{2.7}$$

whose inverse $l^{-1} \colon \mathbb{D}_n \longrightarrow \mathbb{H}_n$ is given by the assignment

$$\zeta \mapsto Z = i(\mathbb{1}_n + \zeta)(\mathbb{1}_n - \zeta)^{-1}. \tag{2.8}$$

The topological closure of $\mathbb{D}_n$ is given by $\overline{\mathbb{D}}_n = \{\zeta \in \mathrm{Sym}_n(\mathbb{C}) \,|\, \zeta\bar{\zeta} \leq \mathbb{1}_n\}$. Through Cayley transformation, the symplectic action on $\mathbb{H}_n$ induces an analogous action of $\mathrm{Sp}_n(\mathbb{R})$ on $\mathbb{D}_n$ given by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \zeta = \Big((A - iC)(\zeta + 1) + i(B - iD)(\zeta - 1) \Big) \Big((A + iC)(\zeta + 1) + i(B + iD)(\zeta - 1) \Big)^{-1},$$

which extends to $\overline{\mathbb{D}}_n$ (see [30, page 15]).

Two points $\zeta, \eta \in \overline{\mathbb{D}}_n$ are called equivalent if they can be connected by a finite number of holomorphic curves.

Definition 2.7. A maximal subset in $\overline{\mathbb{D}}_n$ of mutually equivalent points is called a boundary component of $\mathbb{D}_n$.

The space $\overline{\mathbb{D}}_n$ is divided into a disjoint union of boundary components. Moreover, the symplectic action transforms one boundary component to another. Therefore, the division of $\overline{\mathbb{D}}_n$ into boundary components is invariant under the symplectic action.

For an integer $0 \leq j \leq n$, let

$$\mathbb{D}_n^j = \left\{ \begin{pmatrix} \zeta_j & 0 \\ 0 & \mathbb{1}_{n-j} \end{pmatrix} \,\middle|\, \zeta_j \in \mathbb{D}_j \right\} \cong \mathbb{D}_j$$

Then for all $0 \leq j \leq n$, $\mathbb{D}_n^j$ is a boundary component. In particular, $\mathbb{D}_n \cong \mathbb{D}_n^n$ itself is a boundary component. As

$$\overline{\mathbb{D}}_n = \bigcup_{0 \leq j \leq n} \mathrm{Sp}_n(\mathbb{R})\mathbb{D}_n^j,$$

any boundary component $\mathbb{P}$ of $\mathbb{D}_n$ can be realized as $\mathbb{P} = g\mathbb{D}_n^j$ for some $g \in \mathrm{Sp}_n(\mathbb{R})$ and $0 \leq j \leq n$ (see [30, page 17]). Hence, we call $\mathbb{D}_n^j$ a standard boundary component.

As $\mathbb{D}_n^j$ is isomorphic to the bounded realization $\mathbb{D}_j$ of the degree j Siegel upper half-space, we say $\mathbb{P} = g\mathbb{D}_n^j$ is a boundary component of degree j. If $j < n$, we call $\mathbb{P}$ a proper boundary component. For two boundary components $\mathbb{P}^j, \mathbb{P}^k$ of degrees j, k respectively, we write $\mathbb{P}^j < \mathbb{P}^k$ if $\mathbb{P}^j \subset \overline{\mathbb{P}}_k$. In that case there exists $g \in \mathrm{Sp}_n(\mathbb{R})$ such that $g\mathbb{P}^j = \mathbb{D}_n^j$, $g\mathbb{P}^k = \mathbb{D}_n^k$ and $j \leq k$.

Remark 2.8. This result can be extended to the case of a chain of boundary components

$$\mathbb{P}^0 < \ldots < \mathbb{P}^j \ldots < \mathbb{P}^{n-1}$$

where $\mathbb{P}^j$ is of degree $j \in \{0, 1, \ldots, n-1\}$. Then in that case we have a $g \in \mathrm{Sp}_n(\mathbb{R})$ such that $g\mathbb{P}^j = \mathbb{D}_n^j \, (0 \leq j < n)$.

Definition 2.9. Let $\mathbb{P}$ be a boundary component of $\mathbb{D}_n$. Then the group $P(\mathbb{P}) \subsetneq \mathrm{Sp}_n(\mathbb{R})$ defined by

$$P(\mathbb{P}) = \{g \in \mathrm{Sp}_n(\mathbb{R}) \,|\, g\mathbb{P} = \mathbb{P}\}$$

is called the parabolic subgroup of $\mathrm{Sp}_n(\mathbb{R})$ associated to $\mathbb{P}$.

For the standard proper boundary components $\mathbb{P} = \mathbb{D}_n^j \, (0 \leq j < n)$, the groups $P_j := P(\mathbb{D}_n^j)$ has the structure (see [30, page 21])

$$P_j = \left\{ \begin{pmatrix} A' & 0 & B' & \star \\ \star & u & \star & \star \\ C' & 0 & D' & \star \\ 0 & 0 & \star & u^{-t} \end{pmatrix} \,\middle|\, \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} \in \mathrm{Sp}_j(\mathbb{R}), \, u \in \mathrm{GL}_{n-j}(\mathbb{R}) \right\}.$$

For a general boundary component $\mathbb{P}$ of $\overline{\mathbb{D}}_n$, realized as $\mathbb{P} = g\mathbb{D}_n^j$ for some $g \in \mathrm{Sp}_n(\mathbb{R})$ and some standard boundary component $\mathbb{D}_n^j \, (0 \leq j < n)$, the parabolic subgroup $P(\mathbb{P})$ associated to $\mathbb{P}$ can be obtained as $P(\mathbb{P}) = gP_j g^{-1}$.

Definition 2.10. A boundary component $\mathbb{P}$ of $\mathbb{D}_n$ is called rational if the parabolic subgroup $P(\mathbb{P})$ associated to it is defined over $\mathbb{Q}$. The set

$$\mathbb{D}_n^\star := \bigsqcup_{\mathbb{P} \text{ rational}} \mathbb{P} \subsetneq \overline{\mathbb{D}}_n$$

is called the rational closure of $\mathbb{D}_n$.

Remark 2.11. If $\mathbb{P}$ is a rational boundary component of $\mathbb{D}_n$, then there is a $\sigma \in \mathrm{Sp}_n(\mathbb{Z})$, such that $\sigma\mathbb{P} = \mathbb{D}_n^j$ for some standard boundary component $\mathbb{D}_n^j$ $(0 \leq j < n)$.

The boundary components in the Siegel upper-half space $\mathbb{H}_n$ are obtained from the bounded realization $\mathbb{D}_n$ via the inverse Cayley transform. We denote the standard boundary components on $\mathbb{H}_n$ by $\mathbb{H}_n^j := l^{-1}\mathbb{D}_n^j$ $(0 \leq j < n)$. The rational closure $\mathbb{H}_n^\star$ of $\mathbb{H}_n$ is endowed with the cylindrical topology (see [30, page 35]). Under this topology, a sequence

$$Z(\nu) := \begin{pmatrix} Z_{1,1}(\nu) & Z_{1,2}(\nu) \\ Z_{1,2}(\nu) & Z_{2,2}(\nu) \end{pmatrix} \qquad (\nu \in \mathbb{N}_{>0},\ Z(\nu) \in \mathbb{H}_n,\ Z_{1,1}(\nu) \in \mathbb{H}_j)$$

on $\mathbb{H}_n$ converges to a point $Z \in \mathbb{H}_n^j \cong \mathbb{H}_j$ in $\mathbb{H}_n^\star$ if and only if $Z_{1,1}(\nu) \to Z$ in $\mathbb{H}_j$ and $Y_{2,2}(\nu) - Y_{1,2}(\nu)Y_{2,2}(\nu)Y_{1,2}(\nu) \to \infty$ in $\mathbb{H}_{n-j}$. Under the assumption that $Y_{1,2}(\nu)$ is bounded, the latter condition reduces to $Y_{2,2}(\nu) \to \infty$.

In general, for any boundary component $\mathbb{P}$ of $\mathbb{H}_n$, one can show that there exists a one-parameter subgroup $w_\mathbb{P} \colon \mathbb{R} \to G$ such that

$$\lim_{t \to 0} w_\mathbb{P}(t)^{-1} O = O_\mathbb{P},$$

where $O = i\mathbb{1}_n$ is the base point of $\mathbb{H}_n$ and $O_\mathbb{P}$ is the base point of $\mathbb{P}$. For $\mathbb{P} = \mathbb{H}_n^j$, we denote $w_\mathbb{P}$ by w_j, which takes the form

$$w_j(t) = \begin{pmatrix} \mathbb{1}_j & 0 & 0 & 0 \\ 0 & t\mathbb{1}_{n-j} & 0 & 0 \\ 0 & 0 & \mathbb{1}_j & 0 \\ 0 & 0 & 0 & t^{-1}\mathbb{1}_{n-j} \end{pmatrix} \qquad (t \in \mathbb{R} \setminus \{0\}).$$

It is easy to see that in the above sense, $w_j(t)^{-1} i\mathbb{1}_n \to i\mathbb{1}_j \in \mathbb{H}_n^j \cong \mathbb{H}_j$ as $t \to 0$.

The parabolic subgroups $P(\mathbb{P})$ defined in Definition 2.9 can be characterized in terms of $w_\mathbb{P}$ as

$$P(\mathbb{P}) = \{g \in \mathrm{Sp}_n(\mathbb{R}) \mid \lim_{t \to 0} w_\mathbb{P}(t)\, g\, w_\mathbb{P}(t)^{-1} < \infty\}.$$

We define

$$W(\mathbb{P}) = \{g \in \mathrm{Sp}_n(\mathbb{R}) \mid \lim_{t \to 0} w_\mathbb{P}(t)\, g\, w_\mathbb{P}(t)^{-1} = 1\}$$

Remark 2.12. Given $Z \in \mathbb{P}$, if for any sequence $Z(\nu)\,(\nu \in \mathbb{N}_{>0})$ in $\mathbb{H}_n$ such that $Z(\nu) \to Z$ in $\mathbb{H}_n^\star$, we have $gZ(\nu) \to Z$ for some $g \in \mathrm{Sp}_n(\mathbb{R})$, then it is easy to see that $g \in W(\mathbb{P})$.

For $\mathbb{P} = \mathbb{H}_n^j$, we denote $W(\mathbb{P})$ by W_j, which can be shown to be (see [30, page 21])

$$W_j = \left\{ \begin{pmatrix} \mathbb{1}_j & 0 & 0 & Q \\ P^t & \mathbb{1}_{n-j} & Q^t & B \\ 0 & 0 & \mathbb{1}_j & -P \\ 0 & 0 & 0 & \mathbb{1}_{n-j} \end{pmatrix} \middle| Q^t P + B = P^t Q + B^t \right\}.$$

Then, setting $P = L^t$, $Q = H^t$ and $B = LH^t + S_2$, we have

$$W_j = \left\{ \begin{pmatrix} A & 0 \\ 0 & A^{-t} \end{pmatrix} \begin{pmatrix} \mathbb{1}_n & S \\ 0 & \mathbb{1}_n \end{pmatrix} \middle| A = \begin{pmatrix} \mathbb{1}_j & 0 \\ L & \mathbb{1}_{n-j} \end{pmatrix},\ S = \begin{pmatrix} 0 & H^t \\ H & S_2 \end{pmatrix} \right\} \qquad (0 \leq j < n), \tag{2.9}$$

where $L, H \in \mathbb{R}^{(n-j) \times j}$ and $S_2 \in \mathbb{R}^{(n-j) \times (n-j)}$, $S_2 = S_2^t$.

Next, for an arithmetic subgroup $\Gamma \subsetneq \mathrm{Sp}_n(\mathbb{R})$, consider the set $M := \Gamma \backslash \mathbb{H}_n$.

Theorem 2.13. *The quotient $M^\star := \Gamma \backslash \mathbb{H}_n^\star$ endowed with the quotient topology, is a compact Hausdorff space. It contains M as an open everywhere dense subset. $M^\star$ is the finite union of subspaces $M_j := (\Gamma \cap P(\mathbb{P}_j)) \backslash \mathbb{P}_j$, where $\mathbb{P}_j$ runs through a set of representatives of equivalence classes modulo Γ of rational boundary components of $\mathbb{H}_n$. The closure of M_j is the union of M_j and the subspaces M_k of M_j of strictly smaller degree.*

Proof. See [2, Corollary 4.11]. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

The above compactification $M^\star$ of M is called the Satake-Baily-Borel compactification of M. For $\Gamma = \Gamma_n = \mathrm{Sp}_n(\mathbb{Z})$, it takes the form

$$(\Gamma_n \backslash \mathbb{H}_n)^\star = \bigsqcup_{j=0}^{n} (\Gamma_n \cap P_j) \backslash \mathbb{H}_n^j \cong \bigsqcup_{j=0}^{n} \Gamma_j \backslash \mathbb{H}_j. \tag{2.10}$$

Remark 2.14. By the remarks 2.8 and 2.11, the group Γ_n acts transitively on the rational boundary components of $\mathbb{H}_n$. Hence, for any arithmetic subgroup $\Gamma \subsetneq \mathrm{Sp}(n, \mathbb{R})$, we only need to consider $\mathbb{P}_j = \mathbb{H}_n^k \,(0 \le k < n)$ to fully describe the boundary of $M = \Gamma \backslash \mathbb{H}_n$.

2.4 Siegel modular forms

Definition 2.15. A function $f \colon \mathbb{H}_n \to \mathbb{C}$ is called a Siegel modular form of weight κ and degree n with respect to the Siegel modular group $\Gamma_n = \mathrm{Sp}_n(\mathbb{Z})$ if it satisfies the following conditions:

(i) f is holomorphic,

(ii) $f(\gamma Z) = \det(CZ + D)^\kappa f(Z)$ for all $\gamma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma$,

(iii) For every $Y_0 > 0$, the function f is bounded in the region $Y \ge Y_0$.

We denote the space of all such functions by $\mathcal{M}_\kappa^n(\Gamma_n)$. For all $S \in \mathrm{Sym}_n(\mathbb{Z})$, we have $\left(\begin{smallmatrix} \mathbb{1}_n & S \\ 0 & \mathbb{1}_n \end{smallmatrix}\right) \in \Gamma_n$. Then $f \colon \mathbb{H}_n \to \mathbb{C}$ is a holomorphic function satisfying $f(Z + S) = f(Z)$. Therefore, f has a Fourier expansion of the form

$$f(Z) = \sum_{\substack{T \in \mathrm{Sym}_n(\mathbb{Q}) \\ T \text{ half-integral}}} a(T) \exp(2\pi i \, \mathrm{tr}(TZ)),$$

where $T = (t_{j,k})_{1 \le j,k \le n}$ being half-integral implies that $t_{j,j}, 2t_{j,k} \in \mathbb{Z} \,(1 \le j \le k \le n)$. Also, since $U \in \mathrm{GL}(n, \mathbb{Z})$ implies $\left(\begin{smallmatrix} U^t & 0 \\ 0 & U^{-1} \end{smallmatrix}\right) \in \Gamma_n$, the function f satisfies

$$\det(U)^\kappa f(U^t Z U) = f(Z) \qquad (U \in \mathrm{GL}(n, \mathbb{Z})).$$

So, $U \in \mathrm{SL}(n, \mathbb{Z})$ implies that $f(U^t Z U) = f(Z)$. One can show that a holomorphic function $f \colon \mathbb{H}_n \to \mathbb{C}$ satisfying these two transformation behaviours, i.e., $f(Z + S) = f(Z)$ for integral symmetric matrices S and $f(U^t Z U) = f(Z)$ for $U \in \mathrm{SL}(n, \mathbb{Z})$, under the assumption $n \ge 2$, has a Fourier expansion of the form

$$f(Z) = \sum_{\substack{T \in \mathrm{Sym}_n(\mathbb{Q}),\, T > 0 \\ T \text{ half-integral}}} a(T) \exp(2\pi i \, \mathrm{tr}(TZ)). \tag{2.11}$$

In particular, for some $Y_0 > 0$, the function f is bounded in the region $Y \ge Y_0$. Thus, for $\Gamma = \Gamma_n$ and $n > 1$, condition (iii) follows from conditions (i) and (ii). This is the so-called Koecher's principle.

Definition 2.16. Let $f \colon \mathbb{H}_n \to \mathbb{C}$ be a function so that the limit

$$\lim_{t \to \infty} f \begin{pmatrix} Z & 0 \\ 0 & it \end{pmatrix} \qquad (Z \in \mathbb{H}_{n-1})$$

exists. Then we obtain another function $\Phi(f) \colon \mathbb{H}_{n-1} \to \mathbb{C}$ defined by

$$\Phi(f)(Z) := \lim_{t \to \infty} f \begin{pmatrix} Z & 0 \\ 0 & it \end{pmatrix} \qquad (Z \in \mathbb{H}_{n-1}).$$

This operator $\Phi \colon \mathcal{M}_\kappa^n(\Gamma_n) \to \mathcal{M}_\kappa^{n-1}(\Gamma_{n-1})$ is called the Siegel Φ-operator.

Definition 2.17. A Siegel modular form $f \in \mathcal{M}_\kappa^n(\Gamma_n)$ is called a Siegel cusp form if $\Phi(f) = 0$. We denote the space of Siegel cusp forms by $\mathcal{S}_\kappa^n(\Gamma_n)$.

Proposition 2.18. *A Siegel modular form $f \in \mathcal{M}_\kappa^n(\Gamma_n)$ is a Siegel cusp form if and only if in the Fourier expansion* (2.11)*, $a(T) \neq 0$ implies that T is positive definite.*

Proof. See [10, Hilfssatz 3.9]. $\qquad\square$

Proposition 2.19. *Let $f \in \mathcal{S}_\kappa^n(\Gamma_n)$ and let $c > 0$. Then there exist positive numbers c_1 and c_2 such that*

$$|f(Z)| \leq c_1 \exp(-c_2 \operatorname{tr}(Y))$$

for all $Z \in \mathbb{H}_n$, for which Y is Minkowski reduced and $Y \geq c\mathbb{1}_n$.

Proof. See [26, page 57] $\qquad\square$

Next we define Siegel modular forms for arithmetic subgroups.

Definition 2.20. Let $\Gamma \subset \operatorname{Sp}_n(\mathbb{R})$ be an arithmetic subgroup and $\gamma_j \in \operatorname{Sp}_n(\mathbb{Z})$ ($j = 1, \ldots, h$) denote a set of representatives for the left cosets of $\Gamma \cap \operatorname{Sp}_n(\mathbb{Z})$ in $\operatorname{Sp}_n(\mathbb{Z})$. Then, a *Siegel modular form of weight κ and degree n for Γ* is a function $f \colon \mathbb{H}_n \longrightarrow \mathbb{C}$ satisfying the following conditions:

  (i) f is holomorphic;

  (ii) $f(\gamma Z) = \det(CZ + D)^\kappa f(Z)$ for all $\gamma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma$;

  (iii) given $Y_0 \in \operatorname{Sym}_n(\mathbb{R})$ with $Y_0 > 0$, the quantities $\det(C_j Z + D_j)^{-\kappa} f(\gamma_j Z)$ are bounded in the region $\{Z = X + iY \in \mathbb{H}_n \mid Y \geq Y_0\}$ for the set of representatives $\gamma_j = \left(\begin{smallmatrix} A_j & B_j \\ C_j & D_j \end{smallmatrix}\right) \in \operatorname{Sp}_n(\mathbb{Z})$ ($j = 1, \ldots, h$).

We denote the space of all such functions by $\mathcal{M}_\kappa^n(\Gamma)$. Just as in Definition 2.17, a Siegel modular form $f \in \mathcal{M}_\kappa^n(\Gamma)$ with respect to the arithmetic subgroup Γ is called a Siegel cusp forms with respect to Γ if $\Phi(f) = 0$. We denote the space of all such functions by $\mathcal{S}_\kappa^n(\Gamma)$.

For an arithmetic subgroup $\Gamma \subsetneq \operatorname{Sp}_n(\mathbb{R})$, define

$$t(\Gamma) := \left\{ S \in \operatorname{Sym}_n(\mathbb{R}) \ \middle| \ \begin{pmatrix} \mathbb{1}_n & S \\ 0 & \mathbb{1}_n \end{pmatrix} \in \Gamma \right\}.$$

Then $t(\Gamma)$ is commensurable to $t(\Gamma_n)$. Also, since $t(\Gamma_n)$ is commutative, $[t(\Gamma_n) : t(\Gamma_n \cap \Gamma)] < \infty$ implies that there is an $\ell \in \mathbb{N}_{\geq 1}$ such that $\ell t(\Gamma) \subseteq t(\Gamma_n)$. Hence, $f \in \mathcal{M}_\kappa^n(\Gamma)$ satisfies $f(Z + \ell S) = f(Z)$ ($S \in \operatorname{Sym}_n(\mathbb{R})$). Therefore, the function f_ℓ defined by $f_\ell(Z) = f(\ell Z)$ satisfies $f_\ell(Z + S) = f_\ell(Z)$, ($S \in \operatorname{Sym}_n(\mathbb{R})$) and hence has a Fourier expansion

$$f_\ell(Z) = f(\ell Z) = \sum_{\substack{T \in \operatorname{Sym}_n(\mathbb{Q}),\, T > 0 \\ T \text{ half-integral}}} a(T) \exp(2\pi i \operatorname{tr}(TZ)),$$

whence, replacing Z by Z/ℓ, we have a Fourier expansion of f of the form

$$f(Z) = \sum_{\substack{T \in \operatorname{Sym}_n(\mathbb{Q}),\, T > 0 \\ T \text{ half-integral}}} a(T) \exp\left(\frac{2\pi i}{\ell} \operatorname{tr}(TZ) \right). \tag{2.12}$$

Just like in Proposition 2.18, for a Siegel cusp form $f \in \mathcal{S}_\kappa^n(\Gamma)$ for which, the Fourier coefficients $a(T)$ are 0 unless T is positive definite.

Proposition 2.21. *Let $f \in \mathcal{S}_\kappa^n(\Gamma)$ be a Siegel cusp form. Then, for the function*

$$\varphi(Z) := \det(Y)^{\kappa/2} f(Z),$$

$|\varphi(Z)|$ has a maximum in $\mathbb{H}_n$.

Proof. See [10, Bemerkung 6.10]. $\qquad\square$

Both $\mathcal{M}_\kappa^n(\Gamma)$ and $\mathcal{S}_\kappa^n(\Gamma)$ form finite-dimensional vector spaces over $\mathbb{C}$. The space $\mathcal{S}_\kappa^n(\Gamma)$, with the *Petersson inner product* given by

$$\langle f, g \rangle := \int_{\Gamma \backslash \mathbb{H}_n} (\det Y)^\kappa f(Z) \overline{g}(Z) \, \mathrm{d}\mu_n(Z) \quad (f, g \in S_\kappa^n(\Gamma)),$$

becomes a Hermitian inner product space.

2.5   Siegel Maaß forms and $\Delta^{(\kappa)}$

In order to derive sup-norm bounds for cusp forms $f \in \mathcal{S}_\kappa^n(\Gamma)$, one introduces the function

$$\varphi(Z) := \det(Y)^{\kappa/2} f(Z) \quad (Z = X + iY \in \mathbb{H}_n, \, f \in \mathcal{S}_\kappa^n(\Gamma))$$

with transformation behaviour

$$\varphi(\gamma Z) = \det(\mathrm{Im}(\gamma Z))^{\kappa/2} f(\gamma Z) = \left(\frac{\det(CZ + D)}{\det(C\overline{Z} + D)} \right)^{\kappa/2} \varphi(Z), \tag{2.13}$$

for all $\gamma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix} \right) \in \Gamma$. We begin by defining a space $\mathcal{V}_\kappa^n(\Gamma)$ of real-analytic functions on $\mathbb{H}_n$ that transforms like (2.13) with appropriate growth conditions.

Definition 2.22. Let $\Gamma \subset \mathrm{Sp}_n(\mathbb{R})$ be a subgroup commensurable with $\mathrm{Sp}_n(\mathbb{Z})$, i.e., the intersection $\Gamma \cap \mathrm{Sp}_n(\mathbb{Z})$ is a finite index subgroup of Γ as well as of $\mathrm{Sp}_n(\mathbb{Z})$. We let $\gamma_j \in \mathrm{Sp}_n(\mathbb{Z})$ $(j = 1, \ldots, h)$ denote a set of representatives for the left cosets of $\Gamma \cap \mathrm{Sp}_n(\mathbb{Z})$ in $\mathrm{Sp}_n(\mathbb{Z})$. We then let $\mathcal{V}_\kappa^n(\Gamma)$ denote the space of all functions $\varphi \colon \mathbb{H}_n \to \mathbb{C}$ satisfying the following conditions:

(i) φ is real-analytic;

(ii) $\varphi(\gamma Z) = \det(CZ + D)^{\kappa/2} \det(C\overline{Z} + D)^{-\kappa/2} \varphi(Z)$ for all $\gamma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix} \right) \in \Gamma$;

(iii)   given $Y_0 \in \mathrm{Sym}_n(\mathbb{R})$ with $Y_0 > 0$, there exist $M \in \mathbb{R}_{>0}$ and $N \in \mathbb{N}$ such that the inequalities

$$| \det(C_j Z + D_j)^{-\kappa/2} \det(C_j \overline{Z} + D_j)^{\kappa/2} \varphi(\gamma_j Z)| \leq M \, \mathrm{tr}(Y)^N$$

hold in the region $\{Z = X + iY \in \mathbb{H}_n \,|\, Y \geq Y_0\}$ for the set of representatives $\gamma_j = \left(\begin{smallmatrix} A_j & B_j \\ C_j & D_j \end{smallmatrix} \right) \in \mathrm{Sp}_n(\mathbb{Z})$ $(j = 1, \ldots, h)$.

Remark 2.23. For $\varphi \in \mathcal{V}_\kappa^n(\Gamma)$, we set

$$\|\varphi\|^2 := \int_{\Gamma \backslash \mathbb{H}_n} |\varphi(Z)|^2 \, \mathrm{d}\mu_n(Z),$$

whenever it is defined. In this way we obtain the Hilbert space

$$\mathcal{H}_\kappa^n(\Gamma) := \left\{ \varphi \in \mathcal{V}_\kappa^n(\Gamma) \,\middle|\, \|\varphi\| < \infty \right\}$$

equipped with the inner product

$$\langle \varphi, \psi \rangle = \int_{\Gamma \backslash \mathbb{H}_n} \varphi(Z) \overline{\psi}(Z) \, \mathrm{d}\mu_n(Z) \qquad (\varphi, \psi \in \mathcal{H}_\kappa^n(\Gamma)).$$

We note that in order to enable $\|\varphi\| < \infty$, the exponent $N \in \mathbb{N}$ in part (iii) of Definition 2.22 has to be 0.

To compensate for not being holomorphic, the functions of the form $\varphi(Z) = \det(Y)^{\kappa/2} f(Z)$ $(f \in \mathcal{S}_\kappa^n(\Gamma))$ satisfy the property of being eigenfunctions of a certain differential operator introduced by Maaß that is invariant under the transformation behaviour (2.13).

Given $Z = X + iY \in \mathbb{H}_n$, we start by introducing the following symmetric $(n \times n)$-matrices of partial derivatives:

$$
\text{(i)} \quad \left(\frac{\partial}{\partial X} \right)_{j,k} := \frac{1 + \delta_{j,k}}{2} \frac{\partial}{\partial x_{j,k}},
$$

$$
\text{(ii)} \quad \left(\frac{\partial}{\partial Y} \right)_{j,k} := \frac{1 + \delta_{j,k}}{2} \frac{\partial}{\partial y_{j,k}},
$$

$$
\text{(iii)} \quad \frac{\partial}{\partial Z} := \frac{1}{2} \left(\frac{\partial}{\partial X} - i \frac{\partial}{\partial Y} \right),
$$

$$
\text{(iv)} \quad \frac{\partial}{\partial \overline{Z}} := \frac{1}{2} \left(\frac{\partial}{\partial X} + i \frac{\partial}{\partial Y} \right),
$$

where $\delta_{j,k}$ is the Kornecker delta symbol.

Definition 2.24. Given a positive integer κ, the differential operator $\Delta^{(\kappa)}$ given by

$$
\Delta^{(\kappa)} = \text{tr} \left(Y \left(\left(Y \frac{\partial}{\partial X} \right)^t \frac{\partial}{\partial X} + \left(Y \frac{\partial}{\partial Y} \right)^t \frac{\partial}{\partial Y} \right) - i\kappa Y \frac{\partial}{\partial X} \right)
$$

acting on smooth complex valued functions on $\mathbb{H}_n$ is called the Siegel–Maaß Laplacian of weight κ.

By its invariance under the transformation behaviour (2.13), the operator $\Delta^{(\kappa)}$ acts on the Hilbert space $\mathcal{H}_\kappa^n(\Gamma)$ (see [27, Remark 4.6]).

Definition 2.25. Let $\Gamma \subset \text{Sp}_n(\mathbb{R})$ be a subgroup commensurable with $\text{Sp}_n(\mathbb{Z})$. The elements of the Hilbert space $\mathcal{H}_\kappa^n(\Gamma)$ are called *automorphic forms of weight κ and degree n for Γ*. Moreover, if $\varphi \in \mathcal{H}_\kappa^n(\Gamma)$ is an eigenform of $\Delta^{(\kappa)}$, it is called a *Siegel–Maaß form of weight κ and degree n for Γ*.

Theorem 2.26. *Let $\Gamma \subset \text{Sp}_n(\mathbb{R})$ be a subgroup commensurable with $\text{Sp}_n(\mathbb{Z})$ and let $\varphi \in \mathcal{H}_\kappa^n(\Gamma)$ be a Siegel–Maaß form of weight κ and degree n for Γ. Then, if φ is an eigenform of $\Delta^{(\kappa)}$ with eigenvalue λ, we have $\lambda \in \mathbb{R}$ and*

$$
\lambda \geq \frac{n\kappa}{4}(n - \kappa + 1),
$$

with equality attained if and only if the function φ is of the form $\varphi(Z) = \det(Y)^{\kappa/2} f(Z)$ for some Siegel cusp form $f \in \mathcal{S}_\kappa^n(\Gamma)$ of weight κ and degree n for Γ. In other words, there is an isomorphism

$$
\mathcal{S}_\kappa^n(\Gamma) \cong \ker \left(\Delta^{(\kappa)} + \frac{n\kappa}{4}(n - \kappa + 1)\text{id} \right)
$$

of $\mathbb{C}$-vector spaces, induced by the assignment $f \mapsto \det(Y)^{\kappa/2} f$.

Proof. See [27, Corollary 5.4] $\qquad \square$

3 Construction of the heat kernel

To use (1.10) to obtain sup-norm bounds for the quantity $S_\kappa^\Gamma(Z)$, we need to obtain a somewhat explicit form for the heat kernel $K_t^{(\kappa)}$ corresponding to the Siegel–Maaß Laplacian $\Delta^{(\kappa)}$ on $\mathbb{H}_n$. In the theory of harmonic analysis on symmetric spaces, there is a standard way of obtaining the heat kernel K_t corresponding to the Laplace–Beltrami operator Δ from the spherical function on the given symmetric space, from which, one can use a weight-correction technique to obtain the heat kernel $K_t^{(\kappa)}$ corresponding to $\Delta^{(\kappa)}$. Thus, the problem of obtaining a somewhat explicit form for $K_t^{(\kappa)}$ on $\mathbb{H}_n$ translates to the problem of obtaining a somewhat explicit form for the spherical function ϕ_λ on $\text{Sp}_n(\mathbb{R})$. It is difficult to do it directly. Instead, we wield a technique developed by Flensted-Jensen in [9] of obtaining the spherical function ϕ_λ on a real semisimple group by reducing it to obtaining the spherical function Φ_Λ on the corresponding complex semisimple group, which is much simpler.

In the first two subsections, we briefly recall the general theory of spherical functions on a real semisimple group via the Flensted-Jensen reduction. The general reference for these subsections are [20]

and [9]. In the third subsection, we recall from [12] and [13], the general procedure of construction of the heat kernel on a real semisimple group by spherical inversion. Then in the next two subsections we implement this procedure for the special case of the symplectic group. Finally, in the last subsection, we apply a weight-correction procedure to obtain the heat kernel on $\mathbb{H}_n$ corresponding to $\Delta^{(\kappa)}$.

But first, we need to fix some basic notations for this section.

3.1   Notation

Let $\mathfrak{g}$ be a complex semisimple Lie algebra with a Cartan decomposition $\mathfrak{g} = \mathfrak{u} + \mathfrak{p}$ corresponding to a Cartan involution θ on $\mathfrak{g}$. Let $\mathfrak{g}_0$ be a non-compact real form of $\mathfrak{g}$. The Cartan involution θ on $\mathfrak{g}$ restricts to the Cartan involution θ_0 on $\mathfrak{g}_0$. Let $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{p}_0$ be the Cartan decomposition of $\mathfrak{g}_0$ corresponding to the Cartan involution θ_0 on $\mathfrak{g}_0$. Then $\mathfrak{u} = \mathfrak{k}_0 + i\mathfrak{p}_0$ is a compact real form of $\mathfrak{g}$ and $\mathfrak{k} = \mathfrak{k}_0 + i\mathfrak{k}_0$ is a complex subalgebra of $\mathfrak{g}$. Denote by $\mathfrak{a}$ and $\mathfrak{a}_0$ the maximal abelian subspaces of $\mathfrak{p}$ and $\mathfrak{p}_0$, respectively. We note here that $\mathfrak{k}_0$, $\mathfrak{p}_0$ and $\mathfrak{a}_0$ are related to $\mathfrak{k}$, $\mathfrak{u}$, $\mathfrak{p}$ and $\mathfrak{a}$ via $\mathfrak{k}_0 = \mathfrak{k} \cap \mathfrak{g}_0 = \mathfrak{u} \cap \mathfrak{g}_0$, $\mathfrak{p}_0 = \mathfrak{p} \cap \mathfrak{g}_0$ and $\mathfrak{a}_0 = \mathfrak{a} \cap \mathfrak{g}_0$.

The Killing form B_0 on $\mathfrak{g}_0$ is just the restriction of the complex Killing form B' of $\mathfrak{g}$, whereas the Killing form B of $\mathfrak{g}$ as a real Lie algebra, is $2B'$. This means that the Euclidean structures on $\mathfrak{a}_0$, induced by B_0 and B are different. Denote by $\langle \cdot, \cdot \rangle_0$ and $|| \cdot ||_0$ the scalar product and norm induced by B_0 on $\mathfrak{a}_0$ as well as $\langle \cdot, \cdot \rangle$ and $|| \cdot ||$ for the scalar product and norm induced by B on $\mathfrak{p}$. So in particular

$$\|H\|_0^2 = \frac{1}{2} \|H\|^2 \quad \text{for all} \quad H \in \mathfrak{a}_0.$$

By the Killing form identification of $\mathfrak{a}_0$ and $\mathfrak{a}$ with their duals, $\mathfrak{a}_0^\vee$ is embedded in $\mathfrak{a}^\vee$. The Euclidean structures on the spaces $\mathfrak{a}$ and $\mathfrak{p}$ induce Euclidean structures on the dual spaces $\mathfrak{a}^\vee$ and $\mathfrak{p}^\vee$, respectively, by duality. Denote by $\langle \cdot, \cdot \rangle_0$ and $|| \cdot ||_0$ the induced scalar product and norm on $\mathfrak{a}_0^\vee$ as well as by $\langle \cdot, \cdot \rangle$ and $|| \cdot ||$ the induced scalar product and norm on $\mathfrak{p}^\vee$. So in particular

$$\|\lambda\|_0^2 = \frac{1}{2} \|\lambda\|^2 \quad \text{for all} \quad \lambda \in \mathfrak{a}_0^\vee.$$

Let Δ be the root system of the pair $(\mathfrak{g}, \mathfrak{a})$, by which we mean that Δ is the set of restricted roots for the real Lie algebra $\mathfrak{g}$ with respect to the maximal abelian subalgebra $\mathfrak{a}$. Then each root space $\mathfrak{g}^\alpha$ $(\alpha \in \Delta)$ has dimension $m_\alpha = 2$. Let Δ_0 be the restricted root system of the pair $(\mathfrak{g}_0, \mathfrak{a}_0)$. Then $\Delta_0 = \{\alpha|_{\mathfrak{a}_0} \mid \alpha \in \Delta, \alpha|_{\mathfrak{a}_0} \neq 0\}$. Let W and W_0 be the Weyl groups corresponding to the restricted root systems Δ and Δ_0.

Let Δ^+ and Δ_0^+ be choices of positive restricted roots in the restricted root systems $\Delta(\mathfrak{g}, \mathfrak{a})$ and $\Delta_0(\mathfrak{g}_0, \mathfrak{a}_0)$, respectively. Let $\mathfrak{a}^+$ and $\mathfrak{a}_0^+$ be the corresponding choices of positive Weyl chambers in $\mathfrak{a}$ and $\mathfrak{a}_0$, respectively. Let ρ and ρ_0 denote the half-sums (with multiplicity)

$$\rho = \frac{1}{2} \sum_{\alpha \in \Delta^+} m_\alpha \, \alpha = \sum_{\alpha \in \Delta^+} \alpha \quad \text{and} \quad \rho_0 = \frac{1}{2} \sum_{\lambda \in \Delta_0^+} m_\alpha \alpha$$

of the positive restricted roots for $(\mathfrak{g}, \mathfrak{a})$ and $(\mathfrak{g}_0, \mathfrak{a}_0)$, respectively. Similarly, let π and π_0 denote the products of the indivisible positive restricted roots

$$\pi(\lambda) = \prod_{\alpha \in \Delta^+} \langle \alpha, \lambda \rangle \quad (\lambda \in \mathfrak{a}^\vee) \quad \text{and} \quad \pi_0(\lambda) = \prod_{\alpha \in \Delta_0^+} \langle \alpha, \lambda \rangle \quad (\lambda \in \mathfrak{a}_0^\vee), \tag{3.1}$$

respectively.

Denoting $\mathfrak{n} = \sum_{\alpha \in \Delta^+} \mathfrak{g}^\alpha$ and $\mathfrak{n}_0 = \mathfrak{n} \cap \mathfrak{g}_0$, the algebras $\mathfrak{g}$ and $\mathfrak{g}_0$ have Iwasawa decompositions $\mathfrak{g} = \mathfrak{u} + \mathfrak{a} + \mathfrak{n}$ and $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{a}_0 + \mathfrak{n}_0$, respectively.

Let G be a Lie group with Lie algebra $\mathfrak{g}$, and let K, U, A, N, G_0, K_0, A_0 and N_0 be the analytic subgroups corresponding to $\mathfrak{k}$, $\mathfrak{u}$, $\mathfrak{a}$, $\mathfrak{n}$, $\mathfrak{g}_0$, $\mathfrak{k}_0$, $\mathfrak{a}_0$ and $\mathfrak{n}_0$. Corresponding to the algebra level Iwasawa decompositions $\mathfrak{g} = \mathfrak{u} + \mathfrak{a} + \mathfrak{n}$ and $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{a}_0 + \mathfrak{n}_0$ of $\mathfrak{g}$ and $\mathfrak{g}_0$, respectively, the groups G and G_0 have the group level Iwasawa decompositions $G = UAN$ and $G_0 = K_0 A_0 N_0$, so that the mapping $(u, a, n) \mapsto uan$ is a diffeomorphism of $U \times A \times N$ onto G and $K_0 \times A_0 \times N_0$ onto G_0. Let for $g \in G$, $H(g) \in \mathfrak{a}$ be determined by $g \in U \exp(H(g))N$. If $g \in G_0$, then we have $H(g) \in \mathfrak{a}_0$.

The group G_0 also has the polar decomposition $G_0 = K_0 A_0 K_0$, by which, for each $g \in G_0$, there is an $a \in A_0$ such that $g \in K_0 a K_0$. If, for a particular choice of a positive Weyl chamber $\mathfrak{a}_0^+$, we restrict ourselves to $\overline{A_0^+} = \exp(\overline{\mathfrak{a}_0^+})$, then for each $g \in G_0$, the choice of $a \in \overline{A_0^+}$ such that $g \in K_0 a K_0$, is unique. It can be shown that the set $C^\infty(K_0 \backslash G_0 / K_0)$ of K_0-bi-invariant C^∞-functions on G_0, via restriction to A_0, is in bijective correspondence with $C_{W_0}^\infty(A_0)$, the set of W_0-invariant C^∞-functions on A_0.

Similarly, for the complex group G, we have the polar decomposition $G = U \overline{A^+} U$. Furthermore, G also has the Mostow decomposition $G = U \overline{A_0^+} K$, by which, for each $g \in G$, there is a unique $a \in \overline{A_0^+}$ such that $g \in U a K$. The set $C^\infty(U \backslash G / K)$ is in bijective correspondence, via restriction to A_0, with the set $C_{W_0}^\infty(A_0)$ of W_0-invariant C^∞-functions on A_0 (see [9, Theorem 4.1]).

3.2 Spherical functions on G_0/K_0

Consider the Riemannian globally symmetric space G_0/K_0. Let $\pi \colon G_0 \to G_0/K_0$ denote the natural mapping of G_0 onto G_0/K_0 and $o \in G_0/K_0$ denote the point $o = \pi(e)$, where $e \in G_0$ is the neutral element of G_0. If f is any function on G_0/K_0, let $\widetilde{f}$ denote the function $\widetilde{f} = f \circ \pi$ on G. Let $D(G_0)$ denote the set of all left-invariant differential operators on G_0, $D_{K_0}(G_0) \subsetneq D(G_0)$ the subspace of $D(G_0)$ containing left-invariant differential operators on G_0 which are also right-invariant under K_0 and $D(G_0/K_0)$ the algebra of differential operators on G_0/K_0 invariant under all left translations of G_0/K_0 by G_0.

Definition 3.1. A complex-valued function $\phi \in C^\infty(G_0/K_0)$ on G_0/K_0 is called a spherical function on G_0/K_0 if it satisfies the following properties:

(i) $\phi(o) = 1$,

(ii) $D\phi = \lambda_D \, \phi$ for each $D \in D(G_0/K_0)$, where λ_D is a complex number,

(iii) $\phi(k_0 g K_0) = \phi(g K_0)$ for all $g \in G_0$ and $k_0 \in K_0$.

The function $\widetilde{\phi} = \phi \circ \pi$ on G_0 is called a spherical function on G_0 if and only if ϕ is a spherical function on G_0/K_0.

From the above definition it is easy to see that a spherical function $\widetilde{\phi}$ on G_0 is characterized by the following properties:

(i) $\widetilde{\phi}(e) = 1$,

(ii) $D\widetilde{\phi} = \lambda_D \, \widetilde{\phi}$ for each $D \in D_{K_0}(G_0)$, where λ_D is a complex number,

(iii) $\widetilde{\phi}(k_0 g k_0') = \widetilde{\phi}(g)$ for all $g \in G_0$ and all $k_0, k_0' \in K_0$.

As noted in the last subsection, due to the bi-invariance of $\widetilde{\phi}$ under K_0, it suffices to know $\widetilde{\phi} \in C^\infty(K_0 \backslash G_0 / K_0)$ on the Weyl chamber $A_0^+ = \exp(\mathfrak{a}_0^+)$.

Remark 3.2. As the notion of spherical functions on the group G_0 is equivalent to that on the symmetric space G_0/K_0, for convenience, we denote the spherical functions on both G_0 and G_0/K_0 by ϕ.

For a symmetric space G_0/K_0 of non-compact type, Harish-Chandra [18] gave the following characterization of spherical functions on G_0/K_0 in terms of an integral.

Theorem 3.3. *Let G_0 be a connected semisimple Lie group with finite centre and K_0 a maximal compact subgroup of G_0. Then, as λ runs through $(\mathfrak{a}_0^{\mathbb{C}})^\vee$, the functions*

$$\phi_\lambda(g) = \int_{K_0} \exp\big((i\lambda - \rho_0)(H(gk_0))\big) \, \mathrm{d}\mu(k_0) \quad (g \in G_0), \tag{3.2}$$

where $\mathrm{d}\mu(k_0)$ denotes the Haar measure on K_0, exhaust the class of spherical functions on G_0. Moreover, two such functions ϕ_μ and ϕ_λ are identical if and only if $\mu = \sigma\lambda$ for some σ in the Weyl group W_0.

Proof. See [20, Chapter IV, Theorem 4.3]. $\qquad\square$

Lemma 3.4. *Let G_0 be a connected semisimple Lie group with finite centre and K_0 a maximal compact subgroup of G_0. Then, for $a \in A_0^+$, we have*

$$\phi_0(a) \geq \exp(-\rho_0(\log(a))).$$

Proof. Given the positive Weyl chamber $\mathfrak{a}_0^+$, let ${}^+\mathfrak{a}_0$ denote the dual cone defined by

$$^+\mathfrak{a}_0 := \{H \in \mathfrak{a}_0 \mid B(H, H') > 0, \, \forall \, H' \in \mathfrak{a}_0^+\},$$

and let $\overline{{}^+\mathfrak{a}_0}$ denote its closure. Then, by [20, Chapter IV, Lemma 6.5], for $a \in A_0^+$, we have

$$\log(a) - H(ak_0) \in \overline{{}^+\mathfrak{a}} \quad (k_0 \in K_0),$$

which implies that

$$\rho_0(\log(a)) \geq \rho_0(H(ak_0)) \quad (k_0 \in K_0). \tag{3.3}$$

Then, by Harish-Chandra's characterization of ϕ_λ $(\lambda \in \mathfrak{a}_0^\vee)$ in terms of the integral

$$\phi_\lambda(g) = \int_{K_0} \exp\left((i\lambda - \rho_0)(H(gk_0))\right) \mathrm{d}\mu(k_0) \quad (g \in G_0)$$

given in equation (3.2), we have

$$\phi_0(a) = \int_{K_0} \exp\left(-\rho_0(H(ak_0))\right) \mathrm{d}\mu(k_0)$$

$$\geq \exp\left(-\rho_0(\log(a))\right) \int_K \mathrm{d}\mu(k_0) = \exp\left(-\rho_0(\log(a))\right) \quad (a \in A_0^+),$$

thereby proving the lemma. $\square$

Remark 3.5. In [18], Harish-Chandra gave a series expansion of ϕ_λ with leading coefficients $c(\sigma\lambda)$ $(\sigma \in W_0)$. This function, called *Harish-Chandra's c-function*, features prominently in the theory of spherical transforms and was explicitly determined by Gindikin and Karpelevic as a meromorphic function on $(\mathfrak{a}_0^\mathbb{C})^\vee$. In particular, for $G_0 = \mathrm{Sp}_n(\mathbb{R})$, corresponding to the vector $\lambda = \lambda_1 e_1 + \ldots + \lambda_n e_n \in \mathfrak{a}^\vee \cong \mathbb{R}^n$ (see subsection 3.5), Bhanu Murti [3] showed that

$$|c(\lambda)|^{-2} = \frac{1}{\pi^{n^2/2}} \prod_{1 \leq j \leq n} \frac{\lambda_j}{2} \operatorname{th}\left(\frac{\lambda_j}{2}\pi\right) \prod_{1 \leq j < k \leq n} \frac{\lambda_j + \lambda_k}{2} \operatorname{th}\left(\frac{\lambda_j + \lambda_k}{2}\pi\right) \times$$
$$\times \prod_{1 \leq j < k \leq n} \frac{\lambda_j - \lambda_k}{2} \operatorname{th}\left(\frac{\lambda_j - \lambda_k}{2}\pi\right). \tag{3.4}$$

Definition 3.6. Let f be a smooth function on G_0 which is bi-invariant under K_0. The function $\widehat{f} \colon \mathfrak{a}_0^\vee \to \mathbb{C}$ defined by

$$\widehat{f}(\lambda) = \int_G f(g)\phi_{-\lambda}(g) \, \mathrm{d}\mu(g) \quad (\lambda \in \mathfrak{a}_0^\vee), \tag{3.5}$$

is called the *spherical transform* of f at $\lambda \in \mathfrak{a}_0^\vee$.

The next theorem states the crucial inversion formula for the spherical transform.

Theorem 3.7. *For $g = k_1 \exp(H(g))k_2 \in G_0$ $(k_1, k_2 \in K_0, H(g) \in \mathfrak{a}_0)$, define $|g| := B(H(g), H(g))$. Then the spherical L^2-Schwartz space $\mathscr{C}(K_0\backslash G_0/K_0)$ is the space of all functions $f \in C^\infty(K_0\backslash G_0/K_0)$ such that for all $N \in \mathbb{N}_{\geq 1}$ and $D \in D(G_0)$,*

$$\sup_{g \in G_0} (1 + |g|)^N |Df(g)| \, \phi_0(g)^{-1} < \infty.$$

Let $\mathscr{S}(\mathfrak{a}_0^\vee)$ denote the usual Schwartz space on $\mathfrak{a}_0^\vee$ of rapidly decreasing smooth functions and $\mathscr{S}_{W_0}(\mathfrak{a}_0^\vee)$ be the subspace of W_0-invariant elements. Further, let $\mathscr{H}^R(\mathfrak{a}_0^\vee)$ $(R \in \mathbb{R}_{>0})$ denote the set of functions f

on $\mathfrak{a}_0^\vee$ satisfying the criterion that for each $N \in \mathbb{N}_{\geq 1}$, there exists a constant $C_N \in \mathbb{R}_{>0}$, for which the function f satisfies the condition

$$|f(\lambda)| \leq C_N(1 + |\lambda|)^{-N} \exp(R|\operatorname{Im}(\lambda)|) \quad (\lambda \in \mathfrak{a}_0^\vee)$$

and let $\mathscr{H}(\mathfrak{a}_0^\vee) = \bigcup_{R>0} \mathscr{H}^R(\mathfrak{a}_0^\vee)$. Let $\mathscr{H}_{W_0}(\mathfrak{a}_0^\vee)$ and $\mathscr{H}_{W_0}^R(\mathfrak{a}_0^\vee)$ denote the respective subspaces of W_0-invariant elements. Then the following assertions hold:

(i) The spherical transform given by the assignment $f \mapsto \widehat{f}$ induces a bijection of $\mathscr{C}(K_0 \backslash G_0 / K_0)$ onto $\mathscr{S}_{W_0}(\mathfrak{a}_0^\vee)$.

(ii) Restriction of the domain of the above transform to $C_c^\infty(K_0 \backslash G_0 / K_0) \subsetneq \mathscr{C}(K_0 \backslash G_0 / K_0)$ restricts the bijection onto the subspace $\mathscr{H}_{W_0}(\mathfrak{a}_0^\vee) \subsetneq \mathscr{S}_{W_0}(\mathfrak{a}_0^\vee)$.

(iii) For $f \in \mathscr{C}(K_0 \backslash G_0 / K_0)$ and $\widehat{f} \in \mathscr{S}_{W_0}(\mathfrak{a}_0^\vee)$, we have the formula, called the inverse spherical transform, given by

$$f(g) = \int_{\mathfrak{a}_0^\vee} \widehat{f}(\lambda) \phi_\lambda(g) |c(\lambda)|^{-2} \, \mathrm{d}\lambda \quad (g \in G_0),$$

where $\mathrm{d}\lambda$ denotes the Euclidean measure on $\mathfrak{a}_0^\vee / W_0$.

(iv) As $\mathscr{C}(K_0 \backslash G_0 / K_0)$ is dense in $L^2(K_0 \backslash G_0 / K_0)$ and its image $\mathscr{S}_{W_0}(\mathfrak{a}_0^\vee)$ is dense in $L^2(\mathfrak{a}_0^\vee / W_0, |c(\lambda)|^{-2} \, \mathrm{d}\lambda)$, the spherical transform given by the assignment $f \mapsto \widehat{f}$ extends by continuity to an isometry of $L^2(K_0 \backslash G_0 / K_0)$ onto $L^2(\mathfrak{a}_0^\vee / W_0, |c(\lambda)|^{-2} \, \mathrm{d}\lambda)$, thereby giving the equality

$$\int_{G_0} |f(g)|^2 \, \mathrm{d}\mu(g) = \int_{\mathfrak{a}_0^\vee} |\widehat{f}(\lambda)|^2 |c(\lambda)|^{-2} \, \mathrm{d}\lambda.$$

Proof. See [20, Chapter IV, Section 7] and [14, Sections 5 and 6]. □

One should note here that everything said above concerning spherical functions on real semisimple groups G_0 with respect to K_0 applies equally to spherical functions on complex semisimple groups G with respect to U, which is just the special case where $\mathfrak{g}$ has a complex structure. However, using Harishchandra's series expansion of spherical function in Remark 3.5, in case of a complex Lie group G, the spherical function on G takes the following much simpler form.

Theorem 3.8. *Let G be a complex Lie group. Then the spherical function of G corresponding to $\lambda \in \mathfrak{a}^\vee$ is given by*

$$\phi_\lambda(a) = \frac{\pi(\rho)}{\pi(i\lambda)} \frac{\sum_{\sigma \in W} \det(\sigma) \exp(i\sigma\lambda(\log(a)))}{\sum_{\sigma \in W} \det(\sigma) \exp(\sigma\rho(\log(a)))} \quad (a \in A^+),$$

where $\pi(\lambda) = \prod_{\alpha \in \Delta^+} \langle \alpha, \lambda \rangle$. Moreover, the c-function in this case is given by

$$c(\lambda) = \pi(\rho)/\pi(i\lambda).$$

Proof. See [20, Chapter IV, Section 5] □

3.3 Flensted-Jensen reduction

Now consider $\mathfrak{g}$ as a Lie algebra over $\mathbb{R}$. Let $D_R(K \backslash G)$ denote the set of right-invariant differential operators on the coset space $K \backslash G = \{Kg \,|\, g \in G\}$.

Let

$$C^\infty(K \backslash G / U) = \{\phi \in C^\infty(G) \,|\, \phi(kgu) = \phi(g)\}.$$

The main result in [9] is the following theorem that enables us to lift many questions related to the analysis of spherical functions on a real group G_0, to analogous questions concerning the spherical functions on the corresponding complex group G.

Theorem 3.9. *There is a one-to-one correspondence induced by $\phi \mapsto \phi^\eta$ between the set of spherical functions ϕ on G_0/K_0 and the set of functions $\psi = \phi^\eta$ on G satisfying*

(i) $\psi(e) = 1$,

(ii) $D\psi = \lambda_D \psi$ *for all* $D \in D_R(K\backslash G)$, *where λ_D is a complex number,*

(iii) $\psi \in C^\infty(K\backslash G/U)$,

such that

$$\phi(g\Theta(g)^{-1}) = \phi^\eta(g) \quad (g \in G_0),$$

where $\Theta\colon G_0 \to G_0$ is the involutive automorphism of G_0 such that $(\mathrm{d}\Theta)_e = \theta$.

Proof. See [9, Section 5]. $\qquad\qquad\square$

This allows us to identify $C^\infty(K_0\backslash G/K_0)$ with $C^\infty(K\backslash G/U)$ and write ϕ instead of ϕ^η. Let ϕ_λ ($\lambda \in \mathfrak{a}_0^\vee$) denote the spherical functions on G_0/K_0 and Φ_Λ ($\Lambda \in \mathfrak{a}^\vee$) the spherical functions on G/U. If $\lambda \in \mathfrak{a}_0^\vee$, define $\Lambda \in \mathfrak{a}^\vee$ by

$$\Lambda + i\rho = 2(\lambda + i\rho_0).$$

Then we have

$$\Phi_\Lambda(g) = \int_U \phi_\lambda(ug)\,\mathrm{d}\mu(u) \quad (g \in G),$$

where $\mathrm{d}\mu(u)$ is the normalized Haar measure on U. Under this setup, the following theorem enables us to calculate spherical functions on non-compact real Lie groups from spherical functions on the corresponding complex Lie group via an integral transform.

Theorem 3.10. *Let $\mathfrak{g}_0$ be a normal real form of the complex Lie algebra $\mathfrak{g}$. Assume that the Haar measure $\mathrm{d}\mu(k)$ on K is normalized such that on compact groups the total mass is 1 and on non-compact, d-dimensional spaces the measure is $(2\pi)^{-d/2}$ times the volume element so that the Euclidean Fourier transform is an isometry. Then, the spherical functions ϕ_λ on G_0/K_0 and Φ_Λ on G/U are related by the equation*

$$\phi_\lambda(g\Theta(g)^{-1}) = |c(\lambda)|^2 |\pi_0(\lambda)|^2 \int_K \Phi_{2\lambda}(kg)\,\mathrm{d}\mu(k) \quad (\lambda \in \mathfrak{a}_0^\vee). \tag{3.6}$$

In particular, we have

$$|c(\lambda)|^{-2} = |\pi_0(\lambda)|^2 \int_K \Phi_{2\lambda}(k)\,\mathrm{d}\mu(k) \quad (\lambda \in \mathfrak{a}_0^\vee).$$

Proof. See [9, Section 7]. $\qquad\qquad\square$

3.4 Heat kernel on G_0/K_0

Let Δ_X be the Laplace-Beltrami operator on $X = G_0/K_0$ corresponding to the natural metric on X defined by the Killing form B on $\mathfrak{g}_0$. Then Δ_X can be shown to be descending from the Casimir element $\omega \in U(\mathfrak{g}_0)$ of the universal enveloping algebra $U(\mathfrak{g}_0)$, which, subject to a choice of basis $\{X_j\}_{1\le j\le n}$ of $\mathfrak{g}_0$, can be defined as the sum

$$\omega = \sum_{j=1}^n X_j^* X_j\,,$$

where $\{X_j^*\}_{1\le j\le n}$ is the dual basis with respect to the Killing form B on $\mathfrak{g}_0$ (see [20, p. 331]). The spherical function ϕ_λ ($\lambda \in \mathfrak{a}_0^\vee$) is then an eigenfunction of Δ_X with eigenvalue $\lambda_\omega = -(\langle\rho_0, \rho_0\rangle_0 + \langle\lambda, \lambda\rangle_0)$ (see [20, p. 427]), i.e.,

$$\Delta_X \phi_\lambda(x) = \lambda_\omega\, \phi_\lambda(x) \quad (x \in G_0/K_0,\ \lambda \in \mathfrak{a}_0^\vee).$$

Definition 3.11. The heat kernel on G_0/K_0 is the fundamental solution $K_t(x) \in L^2(G_0/K_0)$ for each $t > 0$ to the heat equation

$$\frac{\partial u_t(x)}{\partial t} = \Delta_X u_t(x) \quad (t > 0,\, x \in G_0/K_0)$$

$$u_0(x) = f(x) \qquad (f \in C_c^\infty) \tag{3.7}$$

in the sense that for any $f \in C_c^\infty$, its convolution $u_t = f * K_t$ is a solution to the above equation satisfying $\|f * K_t\|_2 \to 0$ as $t \to 0$.

In [12, 13], Gangolli, using spherical transform, constructs a function K_t that has the standard properties of the fundamental solution of the heat equation on G_0/K_0.

Theorem 3.12. *Let $\mathscr{C}(K_0\backslash G_0/K_0)$ be the L^2-Schwartz space defined in Theorem 3.7. The function $K_t \colon G_0/K_0 \to \mathbb{R}$ $(t > 0)$ defined by*

$$K_t(x) = \int_{\mathfrak{a}^\vee} \exp(\lambda_\omega\, t)\, \phi_\lambda(x)\, |c(\lambda)|^{-2}\, \mathrm{d}\lambda. \tag{3.8}$$

satisfy the following properties:

(a) $K_t \in \mathscr{C}(K_0\backslash G_0/K_0)$ for each $t > 0$.

(b) $\widehat{K_t}(\lambda) = \exp(\lambda_\omega\, t)$ for all $\lambda \in \mathfrak{a}^\vee$.

(c) $K_t * K_s = K_{t+s}$ for all $t, s > 0$.

(d) For any $f \in C_c^\infty$, $f * K_t$ is a solution to the equation $\partial/\partial t = \Delta_X$ and $\|f * K_t\|_2 \to 0$ as $t \to 0$.

Proof. One obtains (b) by taking a spherical transform of the heat equation (3.7). Then (3.8) is obtained by taking an inverse spherical transform of (b). For further details on the proof, see [12, Proposition 3.1] and [13, Theorem 1]. $\square$

3.5 Spherical function on $\mathbb{H}_n$

In this section, we obtain the spherical function on $\mathrm{Sp}_n(\mathbb{R})/U_n(\mathbb{R}) \cong \mathbb{H}_n$ by using the general procedure for obtaining spherical functions on Riemannian symmetric spaces via the Flensted-Jensen reduction established in subsection 3.3. But first we need to specialize the general notation in subsection 3.1 for the symplectic group in order to have a more explicit structure for these groups and algebras that in turn is essential for obtaining a more explicit structure for the spherical function and the heat kernel in this particular case.

The Lie algebras shall as usual be denoted by gothic letters. Let

$$\mathfrak{g}_0 = \left\{ \begin{pmatrix} A & B \\ C & -A^t \end{pmatrix} \;\middle|\; A, B, C \in \mathbb{R}^{n\times n},\, B = B^t,\, C = C^t \right\}$$

denote the real symplectic algebra $\mathfrak{sp}_n(\mathbb{R})$, while $\mathfrak{g}$ shall denote the complex symplectic algebra $\mathfrak{sp}_n(\mathbb{C}) = \mathfrak{g}_0 + i\mathfrak{g}_0$. On $\mathfrak{g}$, we have the Cartan involution $\theta(X) = -\overline{X}^t$ $(X \in \mathfrak{g})$, which restricts to $\theta_0(X) = -X^t$ $(X \in \mathfrak{g}_0)$ on $\mathfrak{g}_0$. Accordingly, we have the Cartan decomposition $\mathfrak{g} = \mathfrak{u} + \mathfrak{p}$ of $\mathfrak{g}$ into the $(+1)$-eigenspace

$$\mathfrak{u} = \left\{ \begin{pmatrix} A & B \\ -\overline{B} & \overline{A} \end{pmatrix} \;\middle|\; A, B \in \mathbb{C}^{n\times n},\, A = -\overline{A}^t,\, B = B^t \right\}.$$

and the (-1)-eigenspace

$$\mathfrak{p} = \left\{ \begin{pmatrix} A & B \\ \overline{B} & -\overline{A} \end{pmatrix} \;\middle|\; A, B \in \mathbb{C}^{n\times n},\, A = \overline{A}^t,\, B = B^t \right\}.$$

of θ. Similarly, we have the Cartan decomposition $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{p}_0$ of $\mathfrak{g}_0$ into the $(+1)$-eigenspace

$$\mathfrak{k}_0 = \left\{ \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \;\middle|\; A, B \in \mathbb{R}^{n\times n},\, A = -A^t,\, B = B^t \right\},$$

and the (-1)-eigenspace

$$\mathfrak{p}_0 = \left\{ \begin{pmatrix} A & B \\ B & -A \end{pmatrix} \ \middle| \ A, B \in \mathbb{R}^{n \times n}, \ B = B^t, \ A = A^t \right\}$$

of θ_0.

Remark 3.13. We note here that $\mathfrak{u}$ is the symplectic unitary algebra $\mathfrak{u} = \mathfrak{sp}_n(\mathbb{C}) \cap \mathfrak{u}_{2n}$ while $\mathfrak{k}_0$ is the symplectic real orthogonal algebra $\mathfrak{k}_0 = \mathfrak{sp}_n(\mathbb{R}) \cap \mathfrak{o}_{2n}(\mathbb{R})$. The subalgebra $\mathfrak{k} = \mathfrak{k}_0 + i\mathfrak{k}_0$ is then given by the symplectic complex orthogonal algebra $\mathfrak{k} = \mathfrak{sp}_n(\mathbb{C}) \cap \mathfrak{o}_{2n}(\mathbb{C})$.

The maximal abelian subspaces $\mathfrak{a}$ and $\mathfrak{a}_0$ of $\mathfrak{p}$ and $\mathfrak{p}_0$, respectively, are given by the diagonal elements in $\mathfrak{p}$ and $\mathfrak{p}_0$, respectively. As the elements of $\mathfrak{p} = \{X \in \mathfrak{g} \,|\, X = \overline{X}^t\}$ are Hermitian and the elements of $\mathfrak{p}_0 = \{X \in \mathfrak{g}_0 \,|\, X = X^t\}$ are symmetric, the diagonal entries in both $\mathfrak{p}$ and $\mathfrak{p}_0$ are real. Therefore, we have

$$\mathfrak{a} = \mathfrak{a}_0 = \left\{ r = \begin{pmatrix} R & 0 \\ 0 & -R \end{pmatrix} \ \middle| \ R = \begin{pmatrix} r_1 & & 0 \\ & \ddots & \\ 0 & & r_n \end{pmatrix}, \ r_j \in \mathbb{R}, \ 1 \le j \le n \right\}.$$

. As $\mathfrak{a} = \mathfrak{a}_0$, we drop the distinction and from here on denote both by $\mathfrak{a}$.

The basis of the dual space $\mathfrak{a}^\vee$ shall be denoted by $\{e_1, e_2, \ldots, e_n\}$ such that $e_j(r) = r_j$ ($r \in \mathfrak{a}$, $1 \le j \le n$). The generic element of $\mathfrak{a}^\vee$ shall be denoted by $\lambda = \lambda_1 e_1 + \ldots + \lambda_n e_n$ ($\lambda_j \in \mathbb{R}$, $1 \le j \le n$). The Killing form on $\mathfrak{g}$ is given by

$$B(X, Y) = 2(n+1) \operatorname{tr}(XY).$$

Let $E_{j,k}$ ($1 \le j, k \le 2n$) denote the $(2n \times 2n)$-matrix with entry 1 where the j-th row and the k-th column meet, all other entries being 0. Using this Killing form, we can assign to each basis vector e_j ($1 \le j \le n$) in $\mathfrak{a}^\vee$ an element

$$H_j = \frac{1}{4(n+1)} (E_{j,j} - E_{n+j,n+j}) \in \mathfrak{g} \qquad (1 \le j \le n),$$

so that $B(H_j, r) = e_j(r)$. This assignment induces a scalar product on $\mathfrak{a}^\vee$ defined by

$$\langle e_j, e_k \rangle := B(H_j, H_k) = \frac{\delta_{j,k}}{4(n+1)}.$$

The roots of $\mathfrak{g}$ corresponding to $\mathfrak{a}$ are given by

$$\Delta = \{\pm 2e_j \mid 1 \le j \le n\} \cup \{\pm e_j \pm e_k \mid 1 \le j < k \le n\},$$

with each root space $\mathfrak{g}^\alpha$ ($\alpha \in \Delta$) having real dimension (and hence root multiplicity) $m_\alpha = 2$. As $\mathfrak{a} = \mathfrak{a}_0$, for roots of $\mathfrak{g}_0$ corresponding to $\mathfrak{a}_0$, we have $\Delta_0 = \Delta$. However, in case of the real algebra $\mathfrak{g}_0$, each root space $\mathfrak{g}_0^\alpha$ ($\alpha \in \Delta_0$) has real dimension 1. The Weyl group $W = W_0$ consists of the permutations $\sigma : \mathfrak{a} \to \mathfrak{a}$ of elements $r \in \mathfrak{a}$, i.e.,

$$\sigma(r) = \sigma \begin{pmatrix} R & 0 \\ 0 & -R \end{pmatrix} = \begin{pmatrix} \sigma(R) & 0 \\ 0 & -\sigma(R) \end{pmatrix} \quad \text{where} \quad \sigma(R) = \sigma \begin{pmatrix} r_1 & & 0 \\ & \ddots & \\ 0 & & r_n \end{pmatrix} = \begin{pmatrix} \pm r_{\tau(1)} & & 0 \\ & \ddots & \\ 0 & & \pm r_{\tau(n)} \end{pmatrix} \quad (\tau \in S_n).$$

Consequently, we have $W = \mathbb{Z}/n\mathbb{Z} \times S_n$.

The canonical choice of positive roots in Δ is given by

$$\Delta^+ = \{2e_j \mid 1 \le j \le n\} \cup \{e_j + e_k \mid 1 \le j < k \le n\} \cup \{e_j - e_k \mid 1 \le j < k \le n\}. \tag{3.9}$$

Thus, the half-root sum $\rho_0 = 1/2 \sum_{\alpha \in \Delta^+} m_\alpha \alpha$ in the real algebra $\mathfrak{g}_0$ is given by

$$\rho_0 = n\, e_1 + (n-1)\, e_2 + \ldots + (n-j+1)\, e_j + \ldots + 2\, e_{n-1} + e_n$$

and the half-root sum ρ in the complex algebra $\mathfrak{g}$ is given by $\rho = 2\rho_0$. Corresponding to the choice (3.9) of positive roots, the positive Weyl chamber $\mathfrak{a}^+$ of $\mathfrak{a}$ is given by

$$\mathfrak{a}^+ = \left\{ r = \begin{pmatrix} R & 0 \\ 0 & -R \end{pmatrix} \ \middle| \ R = \begin{pmatrix} r_1 & & 0 \\ & \ddots & \\ 0 & & r_n \end{pmatrix}, \ r_1 \ge \ldots \ge r_j \ge \ldots r_n \ge 0 \right\}$$

and the nilpotent algebra $\mathfrak{n} = \sum_{\alpha \in \Delta^+} \mathfrak{g}^\alpha$ is given by

$$\mathfrak{n} = \left\{ \begin{pmatrix} P & Q \\ 0 & -P^t \end{pmatrix} \,\middle|\, P, Q \in \mathbb{C}^{n \times n}, \, P \text{ strictly upper-triangular}, \, Q \text{ symmetric} \right\},$$

wherefrom $\mathfrak{n}_0 = \sum_{\alpha \in \Delta_0^+} \mathfrak{g}_0^\alpha$ can be obtained by restricting to $\mathfrak{g}_0$.

Groups shall as usual continue to be denoted by capital Roman letters. In particular, G_0 shall denote the real symplectic group $\mathrm{Sp}_n(\mathbb{R})$, while G shall denote the complex symplectic group $\mathrm{Sp}_n(\mathbb{C})$. By Remark 3.13, the subgroup $K_0 = \exp(\mathfrak{k}_0) \subsetneq G_0$ is given by the real orthogonal subgroup $\mathrm{Sp}_n(\mathbb{R}) \cap O_{2n}(\mathbb{R})$ of $\mathrm{Sp}_n(\mathbb{R})$, while $K = \exp(\mathfrak{k}) \subsetneq G$ is given by the complex orthogonal group $\mathrm{Sp}_n(\mathbb{C}) \cap O_{2n}(\mathbb{C})$ of $\mathrm{Sp}_n(\mathbb{C})$. Group elements, i.e., the matrices in the matrix groups shall continue to be denoted by small Roman letters. The scalar entries of the matrices shall also be denoted by small letters, while matrix-blocks in matrices written in a block-matrix format shall be denoted by capital letters.

Both real and complex symplectic orthogonal matrices have the same structure

$$k = \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \qquad (AA^t + BB^t = \mathbb{1}_n, \, AB^t = BA^t),$$

but while for $k \in K_0$ this implies that the matrix $A + iB$ is an $(n \times n)$-unitary matrix, no such implication is possible in case of complex orthogonal symplectic matrices $k \in K \setminus K_0$. However, any complex orthogonal matrix $k \in K$ can be represented as $k = k_0 \, k_h$, where $k_0 \in K_0$ is a real orthogonal matrix and k_h is a Hermitian orthogonal matrix (see [15, Theorem 1]). Therefore, a general $k \in K$ can be represented by

$$k = \begin{pmatrix} A_0 & B_0 \\ -B_0 & A_0 \end{pmatrix} \begin{pmatrix} A & B \\ -B & A \end{pmatrix}$$

such that $A_0 + iB_0$ is $(n \times n)$-unitary and $A + iB$ is $(n \times n)$-Hermitian.

The group $A = \exp(\mathfrak{a})$ is given by the group of real diagonal symplectic matrices

$$A = \left\{ \exp(r) = \begin{pmatrix} \exp(R) & 0 \\ 0 & \exp(-R) \end{pmatrix} \,\middle|\, R = \begin{pmatrix} r_1 & & 0 \\ & \ddots & \\ 0 & & r_n \end{pmatrix}, \, r_j \in \mathbb{R}, \, 1 \le j \le n \right\}. \qquad (3.10)$$

By Remark 3.13, the group $U = \exp(\mathfrak{u}) \subsetneq G$ is given by the unitary subgroup $U = \mathrm{Sp}_n(\mathbb{C}) \cap U_{2n} = Sp(n)$ of $\mathrm{Sp}_n(\mathbb{C})$, whose elements can be shown to have the structure

$$U = \left\{ \begin{pmatrix} A & B \\ -\overline{B} & \overline{A} \end{pmatrix} \,\middle|\, A, B \in \mathbb{C}^{n \times n}, A\overline{A}^t + B\overline{B}^t = \mathbb{1}_n, \, AB^t = BA^t \right\}.$$

The group $N = \exp(\mathfrak{n}) \subsetneq G$ is given by

$$N = \left\{ \begin{pmatrix} P & Q \\ 0 & P^{-t} \end{pmatrix} \,\middle|\, P, Q \in \mathbb{C}^{n \times n}, \, P \text{ unit upper-triangular}, \, PQ^t = QP^t \right\},$$

wherefrom $N_0 = \exp(\mathfrak{n}_0) \subsetneq G_0$ can be obtained by restricting to G_0.

The Haar measures of the groups shall be denoted by $\mathrm{d}\mu(x)$, while the Euclidean measures shall be denoted by $\mathrm{d}x$.

This prepares the setup needed to compute the spherical function on $\mathrm{Sp}_n(\mathbb{C})$ corresponding to $\lambda \in \mathfrak{a}^\vee$ using the formula

$$\Phi_\lambda(\exp(r)) = \frac{\pi(\rho)}{\pi(i\lambda)} \frac{\sum_{\sigma \in W} \det(\sigma) \exp(i\sigma\lambda(r))}{\sum_{\sigma \in W} \det(\sigma) \exp(\sigma\rho(r))} \qquad (3.11)$$

in equation (3.13). The quantity

$$\pi(\rho) = \prod_{\alpha \in \Delta^+} \langle 2n \, e_1 + \ldots + 2(n - j + 1) \, e_j + \ldots + 2e_n \, , \, \alpha \rangle$$

will come out to be a positive real constant depending only on n. Since we are not interested in the exact nature of this dependence and keeping track of these constants soon get quite tedious, we shall club all

such constants which are not crucial to our calculation under the generic symbol c_n, which should be interpreted as a positive real constant depending only on n.

The quantity $\pi(i\lambda)$ is of the form

$$\pi(i\lambda) = i^{n^2} \prod_{1 \leq j \leq n} \frac{2\lambda_j}{4(n+1)} \prod_{1 \leq j < k \leq n} \frac{\lambda_j + \lambda_k}{4(n+1)} \prod_{1 \leq j < k \leq n} \frac{\lambda_j - \lambda_k}{4(n+1)}.$$

The above homogeneous polynomial of degree n^2 plays a crucial role in our analysis. Let us formally denote it by

$$\varepsilon(\lambda) := \varepsilon(\lambda_1, \lambda_2, \ldots, \lambda_n) = \prod_{1 \leq j \leq n} \lambda_j \prod_{1 \leq j < k \leq n} (\lambda_j + \lambda_k) \prod_{1 \leq j < k \leq n} (\lambda_j - \lambda_k).$$

Under transpositions $\sigma_{j,k} \in W$ given by the assignment

$$(\lambda_1, .., \lambda_j, .., \lambda_k, .., \lambda_n) \mapsto (\lambda_1, .., \lambda_k, .., \lambda_j, .., \lambda_n)$$

and sign-changes $\sigma_j \in W$ given by the assignment

$$(\lambda_1, .., \lambda_j, .., \lambda_n) \mapsto (\lambda_1, .., -\lambda_j, .., \lambda_n),$$

we have $\sigma_{j,k}((\lambda_j + \lambda_k)(\lambda_j - \lambda_k)) = -(\lambda_j + \lambda_k)(\lambda_j - \lambda_k)$ and $\sigma_j((\lambda_j + \lambda_k)(\lambda_j - \lambda_k)) = (\lambda_j + \lambda_k)(\lambda_j - \lambda_k)$, respectively. Thus, in both the cases we have $\varepsilon(\sigma_{j,k}(\lambda)) = -\varepsilon(\lambda)$ and $\varepsilon(\sigma_j(\lambda)) = -\varepsilon(\lambda)$. Now, since the Weyl group W is generated by these transpositions and sign-changes, the polynomial ε for any formal variable $\lambda = (\lambda_1, \ldots, \lambda_n)$ satisfies the property

$$\varepsilon(\sigma(\lambda)) = \det(\sigma)\varepsilon(\sigma(\lambda)) \qquad (\sigma \in W). \tag{3.12}$$

Next we consider the denominator $\sum_{\sigma \in W} \det(\sigma) \exp(\sigma\rho(r))$ in the right-hand side of the equation (3.11). As $\rho = \sum_{\alpha \in \Delta^+} \alpha$, this denominator can be expressed in the form

$$\sum_{\sigma \in W} \det(\sigma) \exp(\sigma\rho(r)) = \prod_{\alpha \in \Delta^+} (\exp(\alpha(r)) - \exp(-\alpha(r))),$$

whence we obtain

$$\sum_{\sigma \in W} \det(\sigma) \exp(\sigma\rho(r)) = 2^{n^2} \prod_{1 \leq j \leq n} \mathrm{sh}(2r_j) \prod_{1 \leq j < k \leq n} \mathrm{sh}(r_j + r_k) \prod_{1 \leq j < k \leq n} \mathrm{sh}(r_j - r_k).$$

The product of the sh's on the right-hand side of the formula would also play an important role in our later analysis. We formally denote it as

$$\delta(r) := \prod_{1 \leq j \leq n} \mathrm{sh}(r_j) \prod_{1 \leq j < k \leq n} \mathrm{sh}\left(\frac{r_j + r_k}{2}\right) \prod_{1 \leq j < k \leq n} \mathrm{sh}\left(\frac{r_j - r_k}{2}\right).$$

This gives us the formula

$$\Phi_\lambda(\exp(r)) = \frac{c_n}{i^{n^2}} \frac{\sum_{\sigma \in W} \det(\sigma) \exp(i\sigma\lambda(r))}{\varepsilon(\lambda)\,\delta(2r)} \tag{3.13}$$

for the spherical function on $\mathrm{Sp}_n(\mathbb{C})$ corresponding to $\lambda \in \mathfrak{a}^\vee$ at $\exp(r) \in A$. An interesting limiting case of this formula is to determine the spherical function on $\mathrm{Sp}_n(\mathbb{C})$ corresponding to $\lambda = 0$ at $\exp(r) \in A$, which we consider in the next proposition.

Proposition 3.14. *The spherical function on $\mathrm{Sp}_n(\mathbb{C})$ corresponding to $\lambda = 0$ at $\exp(r) \in A$ is given by the formula*

$$\Phi_0(\exp(r)) = c_n \frac{\varepsilon(r)}{\delta(2r)} = c_n \prod_{1 \leq j \leq n} \frac{r_j}{\mathrm{sh}(2r_j)} \prod_{1 \leq j < k \leq n} \frac{r_j + r_k}{\mathrm{sh}(r_j + r_k)} \prod_{1 \leq j < k \leq n} \frac{r_j - r_k}{\mathrm{sh}(r_j - r_k)},$$

where c_n denotes a constant depending only on n.

Proof. We begin by defining a polynomial differential operator

$$\varepsilon(\partial/\partial\lambda) := \varepsilon(\partial/\partial\lambda_1,\ldots,\partial/\partial\lambda_j,\ldots,\partial/\partial\lambda_n).$$

Since both the numerator and the denominator in the formula

$$\Phi_\lambda(\exp(r)) = \frac{c_n}{i^{n^2}}\frac{\sum_{\sigma\in W}\det(\sigma)\exp(i\sigma\lambda(r))}{\varepsilon(\lambda)\,\delta(2r)}$$

are analytic at $\lambda = 0$, the limit at $\lambda\to 0$ can be calculated as

$$\Phi_0(\exp(r)) = \lim_{\lambda\to 0}\frac{c_n}{i^{n^2}}\frac{\sum_{\sigma\in W}\det(\sigma)\exp(i\sigma\lambda(r))}{\varepsilon(\lambda)\,\delta(2r)}$$

$$= \frac{c_n}{i^{n^2}\,\delta(2r)}\lim_{\lambda\to 0}\frac{\varepsilon(\partial/\partial\lambda)\Big(\sum_{\sigma\in W}\det(\sigma)\exp(i\sigma\lambda(r))\Big)}{\varepsilon(\partial/\partial\lambda)\varepsilon(\lambda)},$$

provided both the derivatives converge at $\lambda = 0$.

Now, since $\partial/\partial\lambda_j(\exp(i\sigma\lambda(r))) = \sigma(i\,r_j)\exp(i\sigma\lambda(r)$, it is easy to see that

$$\varepsilon(\partial/\partial\lambda)\exp(i\sigma\lambda(r)) = \varepsilon(i\sigma(r))\exp(i\sigma\lambda(r)),$$

which, by the property (3.12) of ε, becomes

$$\varepsilon(\partial/\partial\lambda)\exp(i\sigma\lambda(r)) = i^{n^2}\det(\sigma)\,\varepsilon(r)\exp(i\sigma\lambda(r)). \tag{3.14}$$

Thus, the derivative of the numerator by $\varepsilon(\partial/\partial\lambda)$, in the limit $\lambda\to 0$, converges to $i^{n^2}\det(\sigma)^2\varepsilon(r)|W| = i^{n^2}\varepsilon(r)2^n n!$.

For a monomial $\lambda_1^{\alpha_1}\cdots\lambda_j^{\alpha_j}\cdots\lambda_n^{\alpha_n}$ ($\alpha_j\in\mathbb{N}_{\geq 0}$, $\sum_{j=1}^n\alpha_j = n^2$) and a differential operator

$$(\partial/\partial\lambda_1)^{\beta_1}\ldots(\partial/\partial\lambda_j)^{\beta_j}\ldots(\partial/\partial\lambda_n)^{\beta_n} \quad (\beta_j\in\mathbb{N}_{\geq 0},\ \sum_{j=1}^n\beta_j = \sum_{j=1}^n\alpha_j = n^2),$$

we have

$$\left(\frac{\partial^{\beta_1}}{\partial\lambda_1^{\beta_1}}\right)\cdots\left(\frac{\partial^{\beta_j}}{\partial\lambda_j^{\beta_j}}\right)\cdots\left(\frac{\partial^{\beta_n}}{\partial\lambda_n^{\beta_n}}\right)\big(\lambda_1^{\alpha_1}\cdots\lambda_j^{\alpha_j}\cdots\lambda_n^{\alpha_n}\big)$$

$$= \begin{cases}\displaystyle\prod_{j=1}^n\frac{\alpha_j(\alpha_j+1)}{2} & \text{for } \alpha_j = \beta_j \quad \forall 1\leq j\leq n,\\[2mm] 0 & \text{otherwise.}\end{cases}$$

Therefore it is easy to see that $\varepsilon(\partial/\partial\lambda)\varepsilon(\lambda)$ is a constant depending only on n and hence we have the requisite limit stated in the proposition. $\qquad\square$

Next we apply Theorem 3.10 to calculate the spherical function ϕ_λ on $G_0 = \mathrm{Sp}_n(\mathbb{R})$ corresponding to $\lambda\in\mathfrak{a}_0^\vee$ by reducing it to the complex case.

Theorem 3.15. *The spherical function ϕ_λ on $G_0 = \mathrm{Sp}_n(\mathbb{R})$ corresponding to*

$$\lambda = \lambda_1\,e_1 + \ldots + \lambda_j\,e_j + \ldots + \lambda_n\,e_n \in\mathfrak{a}^\vee$$

at $\exp(r) = \left(\begin{smallmatrix}\exp(R) & 0\\ 0 & \exp(-R)\end{smallmatrix}\right)\in A$ *with*

$$R = \begin{pmatrix} r_1 & & 0 \\ & \ddots & \\ 0 & & r_n \end{pmatrix} \qquad (r_j\in\mathbb{R}_{\geq 0},\ 1\leq j\leq n),$$

is given by

$$\phi_\lambda(\exp(r)) = \frac{c_n}{i^{n^2}\,\tau(\lambda)} \int\limits_{k\in K} \frac{\sum_{\sigma\in W} \det(\sigma)\exp\big(i\sigma\lambda(\varrho(r,k))\big)}{\delta(\varrho(r,k))}\,\mathrm{d}\mu(k), \tag{3.15}$$

where $\varrho(r,k)$ is the diagonal matrix $\varrho(r,k) = \begin{pmatrix} P(r,k) & 0 \\ 0 & -P(r,k) \end{pmatrix}$ with

$$P(r,k) = \begin{pmatrix} \varrho_1(r,k) & & 0 \\ & \ddots & \\ 0 & & \varrho_n(r,k) \end{pmatrix} \qquad (\varrho_j(r,k)\in\mathbb{R},\ 1\le j\le n)$$

related to $r = \begin{pmatrix} R & 0 \\ 0 & -R \end{pmatrix}$ via the matrix equality $k\exp(r)\overline{k}^t = u\exp(\varrho)\overline{u}^t$ with $k\in K$ and $u\in U$. The functions τ and δ are given by

$$\tau(\lambda) = \prod_{1\le j\le n}\mathrm{th}\Big(\frac{\lambda_j}{2}\pi\Big) \prod_{1\le j<k\le n}\mathrm{th}\Big(\frac{\lambda_j+\lambda_k}{2}\pi\Big) \prod_{1\le j<k\le n}\mathrm{th}\Big(\frac{\lambda_j-\lambda_k}{2}\pi\Big),$$

$$\delta(\varrho) = \prod_{1\le j\le n}\mathrm{sh}(\varrho_j) \prod_{1\le j<k\le n}\mathrm{sh}\Big(\frac{\varrho_j+\varrho_k}{2}\Big) \prod_{1\le j<k\le n}\mathrm{sh}\Big(\frac{\varrho_j-\varrho_k}{2}\Big),$$

while c_n is a positive real constant depending only on n.

Proof. We begin by putting $x = \exp(r/2)$ in Theorem 3.10, so that $x\theta(x)^{-1} = xx^t = \exp(r)$, and we have

$$\phi_\lambda(\exp(r)) = |c(\lambda)|^2|\pi_0(\lambda)|^2 \int\limits_{k\in K} \Phi_{2\lambda}(k\exp(r/2))\,\mathrm{d}\mu(k) \quad (\lambda\in\mathfrak{a}_0^\vee), \tag{3.16}$$

where $\Phi_{2\lambda}$ is the spherical function on the complex group $G = \mathrm{Sp}_n(\mathbb{C})$ corresponding to $2\lambda\in\underline{\mathfrak{a}^\vee}$. As $k\exp(r/2)\in G$, from the $G = U\overline{A^+}U$ decomposition of the complex group, we obtain $\varrho(r,k)\in\overline{A^+}$ and $u,v\in U$, so that $g = k\exp(r/2) = u\exp(\varrho(r,k)/2)v\in G$. Therefore, we have

$$g\overline{g}^t = k\exp(r)\overline{k}^t = u\exp(\varrho(r,k))\overline{u}^t. \tag{3.17}$$

Furthermore, since the spherical function $\Phi_{2\lambda}\in C^\infty(U\backslash G/U)$ on G is bi-invariant under U, we have

$$\Phi_{2\lambda}(k\exp(r/2)) = \Phi_{2\lambda}(u\exp(\varrho(r,k)/2)v) = \Phi_{2\lambda}(\exp(\varrho(r,k)/2)).$$

Now, we plug in the formula for $\Phi_{2\lambda}(\exp(\varrho(r,k)/2))$ in equation (3.16) using the formula (3.13) to obtain

$$\phi_\lambda(\exp(r)) = c_n\frac{|c(\lambda)|^2|\pi_0(\lambda)|^2}{i^{n^2}\varepsilon(2\lambda)} \int\limits_{k\in K} \frac{\sum_{\sigma\in W}\det(\sigma)\exp(i\sigma 2\lambda(\varrho(r,k)/2))}{\delta(\varrho(r,k))}\,\mathrm{d}\mu(k). \tag{3.18}$$

Now $2\lambda(\varrho(r,k)/2) = \lambda(\varrho(r,k))$, so the numerator inside the integral becomes

$$\sum_{\sigma\in W}\det(\sigma)\exp(i\sigma\lambda(\varrho(r,k))).$$

Next $\varepsilon(2\lambda) = 2^{n^2}\varepsilon(\lambda)$ and the positive real constant 2^{n^2} gets assumed within the generic c_n. Also, from the formula

$$|c(\lambda)|^{-2} = \frac{1}{\pi^{n^2/2}} \prod_{1\le j\le n}\frac{\lambda_j}{2}\,\mathrm{th}\Big(\frac{\lambda_j}{2}\pi\Big) \prod_{1\le j<k\le n}\frac{\lambda_j+\lambda_k}{2}\,\mathrm{th}\Big(\frac{\lambda_j+\lambda_k}{2}\pi\Big)\times$$

$$\times\prod_{1\le j<k\le n}\frac{\lambda_j-\lambda_k}{2}\,\mathrm{th}\Big(\frac{\lambda_j-\lambda_k}{2}\pi\Big)$$

for the Harish-Chandra c-function on $\mathrm{Sp}_n(\mathbb{R})$ due to Bhanu Murti mentioned in subsection 3.2, it clearly follows, that in terms of the special functions ε and τ introduced to make our calculations less cumbersome, the above formula can be simply written as

$$|c(\lambda)|^{-2} = c_n\,\varepsilon(\lambda)\tau(\lambda). \tag{3.19}$$

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Thus, noting that

$$|\pi_0(\lambda)|^2 = \left| \prod_{1 \le j \le n} \frac{2\lambda_j}{4(n+1)} \prod_{1 \le j < k \le n} \frac{\lambda_j + \lambda_k}{4(n+1)} \prod_{1 \le j < k \le n} \frac{\lambda_j - \lambda_k}{4(n+1)} \right|^2 = c_n \, \varepsilon(\lambda)^2,$$

we see that $|c(\lambda)|^2 |\pi_0(\lambda)|^2 = c_n(1/\tau(\lambda))$, which brings the integral in the right-hand side of equation (3.18) to

$$\phi_\lambda(\exp(r)) = \frac{c_n}{i^{n^2} \, \tau(\lambda)} \int_{k \in K} \frac{\sum_{\sigma \in W} \det(\sigma) \exp(i\sigma\lambda(\varrho(r,k)))}{\delta(\varrho(r,k))} \, d\mu(k),$$

thereby proving the theorem. $\qquad\square$

Remark 3.16. Since the real symplectic matrix $\exp(r) = \left(\begin{smallmatrix} \exp(R) & 0 \\ 0 & \exp(-R) \end{smallmatrix} \right) \in A \subsetneq G_0 = \mathrm{Sp}_n(\mathbb{R})$ maps the point $i\mathbb{1}_n \in \mathbb{H}_n$ by symplectic action to $Z = i \exp(2R) \in \mathbb{H}_n$ and $\phi_\lambda(\exp(r)) \in C^\infty(K_0 \backslash G_0 / K_0)$ is a radial function on $\mathbb{H}_n$, the formula for ϕ_λ in the above theorem also gives the formula for spherical function on $\mathbb{H}_n$ at a point $Z = k_0 \, i \exp(2R) \in \mathbb{H}_n$ ($k_0 \in K_0$) and hence is also denoted by $\phi_\lambda(2R)$.

3.6 Heat kernel on $\mathbb{H}_n$

In this section, we obtain the heat kernel on $\mathbb{H}_n$ by following the general procedure established in subsection 3.3. We continue with the notation and the basic setup fixed in subsection 3.5.

We begin by computing the eigenvalue $\lambda_\omega = -(\langle \rho_0, \rho_0 \rangle_0 + \langle \lambda, \lambda \rangle_0)$ for the Casimir operator ω on $G_0 = \mathrm{Sp}_n(\mathbb{R})$. For the basis vectors e_j $(1 \le j \le n)$ in $\mathfrak{a}_0^\vee$, the inner product induced by the Killing form of $\mathfrak{g}_0$ on $\mathfrak{a}_0^\vee$ takes the form

$$\langle e_j, e_k \rangle = \frac{\delta_{j,k}}{4(n+1)} \qquad (1 \le j, k \le n).$$

Then, for the half-root sum

$$\rho_0 = n \, e_1 + (n-1) \, e_2 + \ldots + (n-j+1) \, e_j + \ldots + 2 \, e_{n-1} + e_n$$

in $\mathfrak{g}_0$, the inner product $\langle \rho_0, \rho_0 \rangle_0$ turns out to be

$$\langle \rho_0, \rho_0 \rangle_0 = \frac{1^2 + 2^2 + \ldots + n^2}{4(n+1)}.$$

Similarly, for $\lambda = \lambda_1 e_1 + \lambda_2 e_2 + \ldots + \lambda_n e_n \in \mathfrak{a}_0^\vee$, we have

$$\langle \lambda, \lambda \rangle_0 = \frac{\lambda_1^2 + \lambda_2^2 + \ldots + \lambda_n^2}{4(n+1)}.$$

the Casimir operator ω on $G_0 = \mathrm{Sp}_n(\mathbb{R})$ descends on the Siegel upper half-space $\mathbb{H}_n := \{ Z = X + iY \mid X, Y \in \mathbb{R}^{n \times n}, X = X^t, Y = Y^t, Y > 0 \}$ to the operator

$$\Delta = \frac{1}{(n+1)} \, \mathrm{tr} \left(Y \left(\left(Y \frac{\partial}{\partial X} \right)^t \frac{\partial}{\partial X} + \left(Y \frac{\partial}{\partial Y} \right)^t \frac{\partial}{\partial Y} \right) \right).$$

Traditionally, this factor of $1/(n+1)$ is ignored and the Laplace–Beltrami operator on $\mathbb{H}_n$ is written as

$$\Delta = \mathrm{tr} \left(Y \left(\left(Y \frac{\partial}{\partial X} \right)^t \frac{\partial}{\partial X} + \left(Y \frac{\partial}{\partial Y} \right)^t \frac{\partial}{\partial Y} \right) \right),$$

due to which, we correct the value of λ_ω calculated above by multiplying it with a factor of $(n+1)$, thereby setting

$$\lambda_\omega = -\frac{\sum_{j=1}^n j^2 + \sum_{j=1}^n \lambda_j^2}{4}.$$

Now we are ready to compute the heat kernel K_t on $\mathbb{H}_n$ using Theorem 3.15 and the formula (3.8), which is the subject of the next theorem.

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Theorem 3.17. *The heat kernel K_t at a point $Z = k_0\, i \exp(2R) \in \mathbb{H}_n$ on the Siegel upper half-space with $k_0 \in K_0$ and*

$$R = \begin{pmatrix} r_1 & & 0 \\ & \ddots & \\ 0 & & r_n \end{pmatrix} \qquad (r_j \in \mathbb{R}_{\geq 0},\ 1 \leq j \leq n),$$

is given by

$$K_t(2R) = c_n\, \frac{\exp\big(-\sum_{j=1}^n j^2 t/4\big)}{t^{n^2+n/2}} \int\limits_{k \in K} \frac{\varepsilon(\varrho(r,k)) \exp\big(-\sum_{j=1}^n \varrho_j(r,k)^2/t\big)}{\delta(\varrho(r,k))}\, \mathrm{d}\mu(k),$$

where $\varrho(r,k)$ is the diagonal matrix $\varrho(r,k) = \big(\begin{smallmatrix} P(r,k) & 0 \\ 0 & -P(r,k) \end{smallmatrix}\big)$ with

$$P(r,k) = \begin{pmatrix} \varrho_1(r,k) & & 0 \\ & \ddots & \\ 0 & & \varrho_n(r,k) \end{pmatrix} \qquad (\varrho_j(r,k) \in \mathbb{R},\ 1 \leq j \leq n)$$

related to $r = \big(\begin{smallmatrix} R & 0 \\ 0 & -R \end{smallmatrix}\big)$ via the matrix equality $k \exp(r) \overline{k}^t = u \exp(\varrho) \overline{u}^t$ with $k \in K$ and $u \in U$. The functions ε and δ are given by

$$\varepsilon(\varrho) = \prod_{1 \leq j \leq n} \varrho_j \prod_{1 \leq j < k \leq n} (\varrho_j + \varrho_k) \prod_{1 \leq j < k \leq n} (\varrho_j - \varrho_k),$$

$$\delta(\varrho) = \prod_{1 \leq j \leq n} \mathrm{sh}(\varrho_j) \prod_{1 \leq j < k \leq n} \mathrm{sh}\Big(\frac{\varrho_j + \varrho_k}{2}\Big) \prod_{1 \leq j < k \leq n} \mathrm{sh}\Big(\frac{\varrho_j - \varrho_k}{2}\Big),$$

while c_n is a positive real constant depending only on n.

Proof. In Theorem 3.15, we had calculated the spherical function ϕ_λ corresponding to

$$\lambda = \lambda_1\, e_1 + \ldots + \lambda_j\, e_j + \ldots + \lambda_n\, e_n \in \mathfrak{a}^\vee$$

at $Z = k_0\, i \exp(2R)$ as

$$\phi_\lambda(2R) = \frac{c_n}{i^{n^2}\, \tau(\lambda)} \int\limits_{k \in K} \frac{\sum_{\sigma \in W} \det(\sigma) \exp(i\sigma\lambda(\varrho(r,k)))}{\delta(\varrho(r,k))}\, \mathrm{d}\mu(k),$$

Therefore, using the formula (3.8), we have

$$K_t(2R) = \frac{c_n}{i^{n^2}} \exp\Big(-\sum_{j=1}^n j^2 t/4\Big) \int\limits_{k \in K} \frac{I(\varrho(r,k))}{\delta(\varrho(r,k))}\, \mathrm{d}\mu(k) \tag{3.20}$$

where the function $I(\varrho(r,k))$ given by the integral

$$I(\varrho(r,k)) = \sum_{\sigma \in W} \det(\sigma) \int\limits_{\lambda \in \mathfrak{a}^\vee} \frac{|c(\lambda)|^{-2}}{\tau(\lambda)} \exp\Big(-\sum_{j=1}^n \lambda_j^2 t/4 + i\sigma\lambda(\varrho(r,k))\Big)\, \mathrm{d}\lambda.$$

As we noted in equation (3.19), the quantity $|c(\lambda)|^{-2}/\tau(\lambda)$ is just the polynomial $c_n \varepsilon(\lambda)$, where c_n is a positive real constant depending only on n. So our integral simply becomes

$$I(\varrho(r,k)) = c_n \sum_{\sigma \in W} \det(\sigma) \int\limits_{\lambda \in \mathfrak{a}^\vee} \varepsilon(\lambda) \exp\Big(-\sum_{j=1}^n \lambda_j^2 t/4 + i\sigma\lambda(\varrho(r,k))\Big)\, \mathrm{d}\lambda.$$

Also, as in Proposition 3.14, we had noted in equation (3.14) that for the polynomial differential operator $\varepsilon(\partial/\partial\lambda) := \varepsilon(\partial/\partial\lambda_1, \ldots, \partial/\partial\lambda_j, \ldots, \partial/\partial\lambda_n)$ we have

$$\varepsilon(\partial/\partial\lambda) \exp\big(i\sigma\lambda(\varrho(r,k))\big) = i^{n^2} \det(\sigma)\, \varepsilon(\varrho(r,k)) \exp\big(i\sigma\lambda(\varrho(r,k))\big),$$

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the integral $I(\varrho(r,k))$ reduces to calculating the derivative by $\varepsilon(\partial/\partial\lambda)$ of the integral

$$I_0(\varrho(r,k)) = \sum_{\sigma \in W} \int_{\lambda \in \mathfrak{a}^\vee} \exp\Big(- \sum_{j=1}^{n} \lambda_j^2 t/4 + i\sigma\lambda(\varrho(r,k)) \Big)\, \mathrm{d}\lambda$$

as we have $c_n\, \varepsilon(\partial/\partial\lambda)I_0(\varrho(r,k)) = i^{n^2} I(\varrho(r,k))$. This last integral splits into integrals over the individual λ_j-s $(1 \le j \le n)$ as

$$I_0(\varrho(r,k)) = \sum_{\sigma \in W} \prod_{j=1}^{n} \int_{\lambda_j=-\infty}^{\infty} \exp\big(- \lambda_j^2 t/4 + i\lambda_j\sigma(\varrho(r,k)) \big)\, \mathrm{d}\lambda_j.$$

But as we saw before, these individual integrals over λ_j-s are simply

$$\int_{\lambda_j=-\infty}^{\infty} \exp\big(- \lambda_j^2 t/4 + i\lambda_j\sigma(\varrho(r,k)) \big)\, \mathrm{d}\lambda = \frac{2\sqrt{\pi}\exp\big(- \sigma(\varrho(r,k))^2/t \big)}{\sqrt{t}}.$$

Therefore, their product over $1 \le j \le n$ becomes

$$\prod_{j=1}^{n} \int_{\lambda_j=-\infty}^{\infty} \exp\big(- \lambda_j^2 t/4 + i\lambda_j\sigma(\varrho(r,k)) \big)\, \mathrm{d}\lambda_j = \frac{(2\sqrt{\pi})^n}{t^{n/2}} \exp\big(- \sum_{j=1}^{n} \sigma(\varrho(r,k))^2/t \big).$$

However, as we have $\sum_{j=1}^{n} \sigma(\varrho(r,k))^2 = \sum_{j=1}^{n} \varrho_j(r,k)^2$, the integral $I_0(\varrho(r,k))$ evaluates to give

$$I_0(\varrho(r,k)) = \frac{c_n}{t^{n/2}} \exp\Big(- \sum_{j=1}^{n} \varrho_j(r,k)^2/t \Big).$$

Therefore, we have

$$I(\varrho(r,k)) = \frac{c_n}{i^{n^2}} \varepsilon(\partial/\partial\lambda)I_0(\varrho(r,k)) = \frac{c_n}{i^{n^2} t^{n/2}} \varepsilon(-\varrho(r,k)/t) \exp\Big(- \sum_{j=1}^{n} \varrho_j(r,k)^2/t \Big).$$

Now as ε is a homogeneous polynomial of order n^2, we have

$$I(\varrho(r,k)) = \frac{c_n\, i^{n^2}}{t^{n^2+n/2}} \varepsilon(\varrho(r,k)) \exp\Big(- \sum_{j=1}^{n} \varrho_j(r,k)^2/t \Big).$$

Putting this back to equation (3.20), we have the theorem. $\qquad\square$

3.7 weight-κ correction

We continue with the notation in subsections 3.5 and 3.6. Given a function $f\colon G_0/K_0 \to \mathbb{C}$ and $g \in G_0$ define the function $f^g\colon G_0/K_0 \to \mathbb{C}$ by $f^g(x) := f(g^{-1}x)\,(x \in X = G_0/K_0)$.

As the spherical function on X is supposed to be invariant under the left action $f \mapsto f^g\,(g \in G_0)$ of the elements of K_0 on the functions $f\colon X \to \mathbb{C}$, it is constructed by having an eigenfunction u of the invariant differential operators $D \in D(G_0/K_0)$ acted upon by elements $k_0 \in K_0$ and then integrating over K_0 to produce an eigenfunction $\phi(x) = \int_{k_0 \in K_0} u^{k_0}(x)\, \mathrm{d}\mu(k_0)$ of $D \in D(G_0/K_0)$ that is invariant under the action $f \mapsto f^{k_0}\,(k_0 \in K_0)$ of the elements of K_0. This is called the *method of images* and this is basically how one obtains Harish-Chandra's characterization of the spherical function on the symmetric space G_0/K_0 as the integral (3.2).

However, the action $f \mapsto f^g\,(g \in G_0)$ of the elements of G_0 on the functions $f\colon X \to \mathbb{C}$ of X that we have considered in this process is the one that is normally considered in case of group actions, i.e., the action $f \mapsto f^g\,(g \in G_0)$ of the elements of G_0 on the functions $f\colon X \to \mathbb{C}$, where $f^g\colon X \to \mathbb{C}$ is given by the assignment $x \mapsto f(g^{-1}x)$. One can instead introduce a weight factor, i.e., a function $j\colon G_0 \times X \to \mathbb{C}$ satisfying

$$j(g_1\, g_2, x) = j(g_1, g_2x)\, j(g_2, x) \tag{3.21}$$

and consider the action $f \mapsto f_j^g$ $(h \in G_0)$ of the elements of G_0 on the functions $f \colon X \to \mathbb{C}$, where $f_j^g \colon X \to \mathbb{C}$ is given instead by the assignment $x \mapsto j(g^{-1}, x) f(g^{-1} x)$. Then, to compute a spherical function ϕ_j, i.e., an eigenfunction of the invariant differential operators $D \in D_j(G_0/K_0)$ that is invariant under this action of the elements of K_0, we must have an eigenfunction u_j of $D \in D_j(G_0/K_0)$ acted upon by this action of the elements of K_0 and then integrate over K_0 to produce

$$\phi_j(x) = \int_{k_0 \in K_0} j(k_0^{-1}, x)\, u_j(k_0^{-1} x)\, \mathrm{d}\mu(k_0).$$

In subsection 2.5, we had considered one such weighted action of symplectic matrices due to Maaß and obtained a Laplacian invariant under this action. As we eventually want to construct the heat kernel for this weight-κ Siegel–Maaß Laplacian $\Delta^{(\kappa)}$, in this section we adapt the computation of the spherical function and the heat kernel on $\mathbb{H}_n$ in subsections 3.5 and 3.6 for the weight-κ case.

As introduced in (2.13), the weight-κ action of a real symplectic matrix $g \in G_0 = \mathrm{Sp}_n(\mathbb{R})$ on functions $f \colon \mathbb{H}_n \to \mathbb{C}$ on $\mathbb{H}_n$ is given by

$$f^{g^{-1}}(Z) = j_\kappa(g, Z) f(g\, Z) \quad (g \in G_0,\ Z \in \mathbb{H}_n), \tag{3.22}$$

where the weight-factor $j_\kappa(g, Z)$ is given by

$$j_\kappa(g, Z) = \left(\frac{\det(C\overline{Z} + D)}{\det(CZ + D)} \right)^{\kappa/2} \quad \left(g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}_n(\mathbb{R}),\ Z \in \mathbb{H}_n \right).$$

It is easy to check that the weight-factor j_κ satisfies the property (3.21).

The functions $f \colon \mathbb{H}_n \to \mathbb{C}$ on $\mathbb{H}_n$ can be lifted to functions $\widetilde{f} \colon \mathrm{Sp}_n(\mathbb{R}) \to \mathbb{C}$ defined by

$$\widetilde{f}(g) := j_\kappa(g, i\mathbb{1}_n)\, f(g\, i\mathbb{1}_n).$$

There is a one-to-one correspondence between the functions on $f \colon \mathbb{H}_n \to \mathbb{C}$ on $\mathbb{H}_n$ that satisfy

$$f(Z) = j_\kappa(g', Z) f(g'Z) \quad (Z = g\, i\mathbb{1}_n \in \mathbb{H}_n)$$

for some $g' \in G_0$ and the functions $\widetilde{f} \colon G_0 \to \mathbb{C}$ on G_0 that satisfy

(i) $\widetilde{f}(g'\, g) = \widetilde{f}(g)$ for all $g \in G_0$

(ii) $\widetilde{f}(g\, k_0) = j_\kappa(k_0, i\mathbb{1}_n) \widetilde{f}(g)$ for all $g \in G_0$ and $k_0 \in K_0$.

Therefore, to compute the weight-κ spherical function on $\mathbb{H}_n$, we need to integrate over the action

$$\widetilde{f}^{k_0^{-1}}(g) = j_\kappa(k_0, i\mathbb{1}_n)^{-1} \widetilde{f}(k_0\, g)$$

of K_0 on G_0, which takes the explicit form

$$\widetilde{f}^{k_0^{-1}}(g) = \left(\frac{\det(A + iB)}{\det(A - iB)} \right)^{\kappa/2} \widetilde{f}(k_0\, g) \quad \left(k_0 = \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \in K_0,\ g \in \mathrm{Sp}_n(\mathbb{R}) \right)$$

when we write k_0 in the familiar block-matrix form for symplectic matrices.

However, this lifts the weight-κ action of G_0 on $\mathbb{H}_n$ to G_0, while as our computation of the spherical function, made by reducing it to the complex case, takes place in the complex group G, we need to determine this action in G. As in the complex reduction method, we consider the Lie algebra $\mathfrak{g} = \mathfrak{g}_0 + i\mathfrak{g}_0$ of G as a real Lie algebra, its elements are canonically embedded in the space of $(4n \times 4n)$-real matrices as

$$X \mapsto \begin{pmatrix} \mathrm{Re}(X) & \mathrm{Im}(X) \\ -\mathrm{Im}(X) & \mathrm{Re}(X) \end{pmatrix} \quad (X \in \mathfrak{g}).$$

Therefore, the elements of G are also canonically embedded in the space of $(4n \times 4n)$-real matrices as

$$g = \exp(X) \mapsto \begin{pmatrix} \mathrm{Re}(g) & \mathrm{Im}(g) \\ -\mathrm{Im}(g) & \mathrm{Re}(g) \end{pmatrix} \quad (g \in G).$$

Accordingly, the element $i\mathbb{1}_n \in \mathbb{H}_n$, under this canonical embedding of $(n \times n)$-complex matrices into $(2n \times 2n)$-real matrices takes the form

$$i\mathbb{1}_n \mapsto \begin{pmatrix} 0 & \mathbb{1}_n \\ -\mathbb{1}_n & 0 \end{pmatrix} = J_n.$$

This gives us the weight-κ action of K on the functions $\widetilde{f} \colon G \to \mathbb{C}$ on G as

$$\widetilde{f}^{k^{-1}}(g) = j_\kappa(k, J_n)\widetilde{f}(k\,g) \qquad (k \in K, \ g \in \mathrm{Sp}_n(\mathbb{C})),$$

where the weight-factor j_κ is of the form

$$j_\kappa(k, J_n) = \left(\frac{\det(\mathrm{Re}(k) + \mathrm{Im}(k)J_n)}{\det(\mathrm{Re}(k) - \mathrm{Im}(k)J_n)} \right)^{\kappa/2}$$

Now writing $k \in K$ as $k = k_0\,k_h$, where k_0 is real orthogonal and k_h is Hermitian orthogonal, by the property (3.21) of j_κ, we have $j_\kappa(k, J_n) = j_\kappa(k_0, k_h J_n)j_\kappa(k_h, J_n)$. Since k_0 is real orthogonal, $\mathrm{Im}(k_0) = 0$ so that $j_\kappa(k_0, k_h J_n) = 1$, thereby giving $j_\kappa(k, J_n) = j_\kappa(k_h, J_n)$.

To calculate $j_\kappa(k_h, J_n)$ more explicitly, we need to write k_h in the block-diagonal form

$$k_h = \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \qquad (AA^t + BB^t = \mathbb{1}_n, \ AB^t = BA^t, \ A = \overline{A}^t, \ B = -\overline{B}^t).$$

The matrix $h := A + iB$ is obviously $(n \times n)$-Hermitian. The orthogonality condition $AA^t + BB^t = \mathbb{1}_n$, $AB^t = BA^t$ can be restated as

$$(A + iB)(A^t - iB^t) = \mathbb{1}_n,$$

so that, we have $A - iB = h^{-t}$. As h is Hermitian, this also implies that $A - iB = \overline{h}^{-1}$. Then to calculate $\det(\mathrm{Re}(k) + \mathrm{Im}(k)J_n)$ explicitly, we have

$$\det(\mathrm{Re}(k) + \mathrm{Im}(k)J_n)$$

$$= \det\left[\begin{pmatrix} \frac{1}{2}(A + \overline{A}) & \frac{1}{2}(B + \overline{B}) \\ -\frac{1}{2}(B + \overline{B}) & \frac{1}{2}(A + \overline{A}) \end{pmatrix} + \begin{pmatrix} -\frac{i}{2}(A - \overline{A}) & -\frac{i}{2}(B - \overline{B}) \\ \frac{i}{2}(B - \overline{B}) & -\frac{i}{2}(A - \overline{A}) \end{pmatrix} \begin{pmatrix} 0 & \mathbb{1}_n \\ -\mathbb{1}_n & 0 \end{pmatrix} \right]$$

$$= \det\left[\begin{pmatrix} \frac{1}{2}((A + iB) + (\overline{A} - i\overline{B})) & -\frac{i}{2}((A + iB) - (\overline{A} - i\overline{B})) \\ \frac{i}{2}((A + iB) - (\overline{A} - i\overline{B})) & \frac{1}{2}((A + iB) + (\overline{A} - i\overline{B})) \end{pmatrix} \right],$$

whence using the relations

$$A + iB = h, \qquad A - iB = \overline{h}^{-1}, \qquad \overline{A} + i\overline{B} = h^{-1}, \qquad \overline{A} - i\overline{B} = \overline{h}, \tag{3.23}$$

it follows that

$$\det(\mathrm{Re}(k) + \mathrm{Im}(k)J_n) = \det\left[\begin{pmatrix} \frac{1}{2}(h + \overline{h}) & -\frac{i}{2}(h - \overline{h}) \\ \frac{i}{2}(h - \overline{h}) & \frac{1}{2}(h + \overline{h}) \end{pmatrix} \right]$$

$$= \det\left[\begin{pmatrix} \mathrm{Re}(h) & \mathrm{Im}(h) \\ -\mathrm{Im}(h) & \mathrm{Re}(h)) \end{pmatrix} \right].$$

Now, since for any two $(n \times n)$ real matrices X, Y, we have

$$\det\left[\begin{pmatrix} X & Y \\ -Y & X \end{pmatrix} \right] = \det(X)\det(X + YX^{-1}Y) = |\det(X + iY)|^2,$$

we have here $\det(\mathrm{Re}(k) + \mathrm{Im}(k)J_n) = |\det(h)|^2$ and similarly, $\det(\mathrm{Re}(k) - \mathrm{Im}(k)J_n) = |\det(h^{-1})|^2$, thereby giving

$$j_\kappa(k, J_n) = j_\kappa(k_h, J_n) = \det(h)^{2\kappa} \qquad \left(k_h = \begin{pmatrix} A & B \\ -B & A \end{pmatrix}, \ h = A + iB, \ h \text{ hermitian} \right).$$

To obtain the weight-κ spherical function on $\mathbb{H}_n$, we need only to multiply the integrand in equation (3.15) in Theorem 3.15 with this weight-factor corresponding to $k \in K$. We restate this result as a theorem for future reference.

Theorem 3.18. *The spherical function $\phi_\lambda^{(\kappa)}$ on the Siegel upper half-space $\mathbb{H}_n$ for the weight-κ action*

$$f^{g^{-1}}(Z) = \left(\frac{\det(C\overline{Z}+D)}{\det(CZ+D)}\right)^{\kappa/2} f(g\,Z) \quad \left(g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}_n(\mathbb{R}),\; Z \in \mathbb{H}_n\right),$$

of the symplectic matrices $g \in \mathrm{Sp}_n(\mathbb{R})$ on the functions $f\colon \mathbb{H}_n \to \mathbb{C}$, corresponding to

$$\lambda = \lambda_1\,e_1 + \ldots + \lambda_j\,e_j + \ldots + \lambda_n\,e_n \in \mathfrak{a}^\vee$$

at $Z = k_0\,i\exp(2R)$ with $k_0 \in K_0$ and

$$R = \begin{pmatrix} r_1 & & 0 \\ & \ddots & \\ 0 & & r_n \end{pmatrix} \qquad (r_j \in \mathbb{R}_{\geq 0},\; 1 \leq j \leq n),$$

is given by

$$\phi_\lambda^{(\kappa)}(2R) = \frac{c_n}{i^{n^2}\,\tau(\lambda)} \int\limits_{k\in K} \frac{\sum_{\sigma\in W} \det(\sigma)\exp(i\sigma\lambda(\varrho(r,k)))}{\delta(\varrho(r,k))}\,\det(h(k))^{2\kappa}\,\mathrm{d}\mu(k), \tag{3.24}$$

where $h(k) = A + iB$ is the Hermitian matrix obtained from the decomposition of $k \in K$ into real orthogonal $k_0 \in K_0$ and Hermitian orthogonal

$$k_h = \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \qquad (AA^t + BB^t = \mathbb{1}_n,\; AB^t = BA^t,\; A = \overline{A}^t,\; B = -\overline{B}^t)$$

as $k = k_0\,k_h$ and $\varrho(r,k)$ is the diagonal matrix $\varrho(r,k) = \begin{pmatrix} P(r,k) & 0 \\ 0 & -P(r,k) \end{pmatrix}$ with

$$P(r,k) = \begin{pmatrix} \varrho_1(r,k) & & 0 \\ & \ddots & \\ 0 & & \varrho_n(r,k) \end{pmatrix} \quad (\varrho_j(r,k) \in \mathbb{R},\; 1 \leq j \leq n)$$

related to $r = \begin{pmatrix} R & 0 \\ 0 & -R \end{pmatrix}$ via the matrix equality $k\exp(r)\overline{k}^t = u\exp(\varrho)\overline{u}^t$ with $k \in K$ and $u \in U$. The functions τ and δ are given by

$$\tau(\lambda) = \prod_{1\leq j\leq n} \mathrm{th}\left(\frac{\lambda_j}{2}\pi\right) \prod_{1\leq j<k\leq n} \mathrm{th}\left(\frac{\lambda_j+\lambda_k}{2}\pi\right) \prod_{1\leq j<k\leq n} \mathrm{th}\left(\frac{\lambda_j-\lambda_k}{2}\pi\right),$$

$$\delta(\varrho) = \prod_{1\leq j\leq n} \mathrm{sh}(\varrho_j) \prod_{1\leq j<k\leq n} \mathrm{sh}\left(\frac{\varrho_j+\varrho_k}{2}\right) \prod_{1\leq j<k\leq n} \mathrm{sh}\left(\frac{\varrho_j-\varrho_k}{2}\right),$$

while c_n is a positive real constant depending only on n.

The Siegel-Maaß Laplacian $\Delta^{(\kappa)}$ is invariant under the weight-κ action (3.22) of the symplectic group. This is due to the fact that the Casimir operator $\omega \in U\mathfrak{g}_0$ descends under this action to the $\Delta^{(\kappa)}$. The only part in our computation of the heat kernel where the action of the group G_0 on functions on G_0/K_0 played a role was in the computation of the spherical function in subsection 3.3, which was done by integrating over the action of the complex orthogonal group K on the spherical function for the complex group G. Therefore, to construct the heat kernel for the Laplacian $\Delta^{(\kappa)}$ on $\mathbb{H}_n$ ($n \in \mathbb{N}_{\geq 1}$), we only need to adapt the formula for the spherical function by suitably altering the action of the group K on the spherical function Φ_Λ for the complex group G, which was done in Theorem 3.18 by multiplying the integrand in equation (3.15) in Theorem 3.15 with a weight-factor $\det(h(k))^{-2\kappa}$, where $h(k) = A + iB$ is the Hermitian matrix obtained from the decomposition of $k \in K$ into real orthogonal $k_0 \in K_0$ and Hermitian orthogonal

$$k_h = \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \qquad (AA^t + BB^t = \mathbb{1}_n,\; AB^t = BA^t,\; A = \overline{A}^t,\; B = -\overline{B}^t)$$

as $k = k_0\,k_h$. To obtain a more explicit bound for the heat kernel on $\mathbb{H}_n$ corresponding to $\Delta^{(\kappa)}$, we need a bound for this factor $\det(h(k))$ in terms of the diagonal matrices ϱ and r, which is what we undertake next.

Lemma 3.19. *Let A be a $(n \times n)$-Hermitian matrix. Let eigenvalues of A be labeled according to increasing size:*

$$\lambda_{\min}(A) = \lambda_1(A) \leq \ldots \leq \lambda_n(A) = \lambda_{\max}(A)$$

Let r be an integer with $1 \leq r \leq n$, and let A_r denote any $(r \times r)$-principal submatrix of A obtained by deleting $n - r$ rows and the corresponding $n - r$ columns from A. For each integer k such that $1 \leq k \leq r$, we have

$$\lambda_k(A) \leq \lambda_k(A_r) \leq \lambda_{k+n-r}(A)$$

Proof. See [31, p. 189, Theorem 4.3.15] □

Theorem 3.20. *Let $k \in K$ be a complex symplectic orthogonal matrix and $h(k) = A + iB$ be the Hermitian matrix obtained from the decomposition of k into real orthogonal $k_0 \in K_0$ and Hermitian orthogonal*

$$k_h = \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \qquad (AA^t + BB^t = \mathbb{1}_n,\ AB^t = BA^t,\ A = \overline{A}^t,\ B = -\overline{B}^t)$$

as $k = k_0 k_h$. Let R be the diagonal matrix

$$R = \begin{pmatrix} r_1 & & 0 \\ & \ddots & \\ 0 & & r_n \end{pmatrix} \qquad (r_j \in \mathbb{R}_{\geq 0},\ 1 \leq j \leq n)$$

and $r = \begin{pmatrix} R & 0 \\ 0 & -R \end{pmatrix}$. Let $u \exp(\varrho)\overline{u}^t = k \exp(r)\overline{k}^t$ be the eigendecomposition of the Hermitian matrix $k \exp(r)\overline{k}^t$ with $u \in U$ unitary symplectic and $\varrho(r,k)$ the diagonal matrix $\varrho(r,k) = \begin{pmatrix} P(r,k) & 0 \\ 0 & -P(r,k) \end{pmatrix}$ with

$$P(r,k) = \begin{pmatrix} \varrho_1(r,k) & & 0 \\ & \ddots & \\ 0 & & \varrho_n(r,k) \end{pmatrix} \qquad (\varrho_j(r,k) \in \mathbb{R},\ 1 \leq j \leq n).$$

Then $\det(h(k))$ is bounded above by

$$\det(h(k)) \leq \frac{\exp(\sum_{j=1}^n |\varrho_j|)}{\prod\limits_{j=1}^n \mathrm{ch}(r_j)}.$$

Proof. Let l be the $(2n \times 2n)$-matrix

$$l = \frac{1-i}{2} \begin{pmatrix} \mathbb{1}_n & -i\mathbb{1}_n \\ \mathbb{1}_n & i\mathbb{1}_n \end{pmatrix}. \tag{3.25}$$

It is easy to check that l is a symplectic unitary matrix, whose inverse is given by

$$l^{-1} = \overline{l}^t = \frac{1+i}{2} \begin{pmatrix} \mathbb{1}_n & \mathbb{1}_n \\ i\mathbb{1}_n & -i\mathbb{1}_n \end{pmatrix}.$$

Also let the symplectic real orthogonal k_0 and Hermitian orthogonal k_h be of the forms

$$k_0 = \begin{pmatrix} A_0 & B_0 \\ -B_0 & A_0 \end{pmatrix} \qquad \text{and} \qquad k_h = \begin{pmatrix} A & B \\ -B & A \end{pmatrix},$$

respectively. Then multiplying the matrix $k \exp(r)\overline{k}^t = k_0 k_h \exp(r) k_h k_0^t$ from the left by l and from the right by l^{-1}, and writing the product as

$$l(k \exp(r)\overline{k}^t)l^{-1} = (l k_0 l^{-1})(l k_h l^{-1})(l \exp(r) l^{-1})(l k_h l^{-1})(l k_0^t l^{-1}),$$

in the block decomposed form, we have

$$l(k \exp(r)\overline{k}^t)l^{-1} = \begin{pmatrix} A_0+iB_0 & 0 \\ 0 & A_0-iB_0 \end{pmatrix}\begin{pmatrix} A+iB & 0 \\ 0 & A-iB \end{pmatrix}\begin{pmatrix} \mathrm{ch}(R) & \mathrm{sh}(R) \\ \mathrm{sh}(R) & \mathrm{ch}(R) \end{pmatrix}\begin{pmatrix} A+iB & 0 \\ 0 & A-iB \end{pmatrix}\begin{pmatrix} A_0^t-iB_0^t & 0 \\ 0 & A_0^t+iB_0^t \end{pmatrix}.$$

Since $k_0 \in K_0$ is real orthogonal, we know that the matrix $w := A_0+iB_0$ is unitary. By the hypothesis of the theorem, $h = A+iB$ is Hermitian. In that case, in equation (3.23), we noted that $A-iB = \overline{h}^{-1} = h^{-t}$. With these notations, the above matrix equation becomes

$$l(k \exp(r)\overline{k}^t)l^{-1} = \begin{pmatrix} w\,h\,\mathrm{ch}(R)\,h\,\overline{w}^t & w\,h\,\mathrm{sh}(R)\,h^{-t}\,w^t \\ \overline{w}h^{-t}\,\mathrm{sh}(R)\,h\,\overline{w}^t & \overline{w}\,h^{-t}\,\mathrm{ch}(R)\,h^{-t}\,w^t \end{pmatrix}.$$

Note, that the determinant of the $(1,1)$-block of the above matrix is $\det(h)^2 \det(\,\mathrm{ch}(R))$.

Now coming to the other side of the matrix equation $u \exp(\varrho)\overline{u}^t = k\exp(r)\overline{k}^t$, as l is symplectic unitary, the matrix $s = l\,u$ is also unitary. Writing s in the block decomposed form

$$s = \begin{pmatrix} A & B \\ -\overline{B} & \overline{A} \end{pmatrix},$$

we write the matrix $l(u\exp(\varrho)\overline{u}^t)l^{-1} = s\exp(\varrho)\overline{s}^t$ as

$$\begin{aligned} s\exp(\varrho)\overline{s}^t &= \begin{pmatrix} A & B \\ -\overline{B} & \overline{A} \end{pmatrix}\begin{pmatrix} \exp(P) & 0 \\ 0 & \exp(-P) \end{pmatrix}\begin{pmatrix} \overline{A}^t & -B^t \\ \overline{B}^t & A^t \end{pmatrix} \\ &= \begin{pmatrix} A\exp(P)\overline{A}^t + B\exp(-P)\overline{B}^t & -A\exp(P)B^t + B\exp(-P)A^t \\ -\overline{B}\exp(P)\overline{A}^t + \overline{A}\exp(-P)\overline{B}^t & \overline{B}\exp(P)B^t + \overline{A}\exp(-P)A^t \end{pmatrix} \end{aligned}$$

Comparing the determinant of the $(1,1)$-block of this matrix with that of $l(k\exp(r)\overline{k}^t)l^{-1}$, we have

$$\det(h)^2 \det(\,\mathrm{ch}(R)) = \det(A\exp(P)\overline{A}^t + B\exp(-P)\overline{B}^t).$$

Let us denote by m the $(2n \times 2n)$-Hermitian matrix $s\exp(\varrho)\overline{s}^t$, and by M its $(n \times n)$-principal submatrix

$$M := A\exp(P)\overline{A}^t + B\exp(-P)\overline{B}^t.$$

Now, m being a $(2n \times 2n)$-Hermitian matrix with eigenvalues $\exp(\pm\varrho_1),\ldots,\exp(\pm\varrho_n)$ and M being the $(n \times n)$-principal submatrix of m, by Lemma 3.19, we have

$$\lambda_k(m) \leq \lambda_k(M) \leq \lambda_{n+k}(m) \qquad (1 \leq k \leq n),$$

which implies

$$\lambda_1(m)\cdots\lambda_n(m) \leq \det(M) \leq \lambda_{n+1}(m)\cdots\lambda_{2n}(m).$$

The n largest eigenvalues of m are $\exp(|\varrho_1|),\ldots,\exp(|\varrho_n|)$, and The n smallest eigenvalues of m are $\exp(-|\varrho_1|),\ldots,\exp(-|\varrho_n|)$. Therefore, we have

$$\lambda_1(A)\cdots\lambda_n(A) = \exp\Big(-\sum_{j=1}^n |\varrho_j|\Big) \quad\text{and}\quad \lambda_{n+1}(A)\cdots\lambda_{2n}(A) = \exp\Big(\sum_{j=1}^n |\varrho_j|\Big),$$

from where it follows that

$$\exp\Big(-\sum_{j=1}^n |\varrho_j|\Big) \leq \det(M) \leq \exp\Big(\sum_{j=1}^n |\varrho_j|\Big),$$

thereby proving the requisite determinant-inequality. □

This of course provides a very useful upper bound on the heat kernel corresponding to the Siegel–Maaß Laplacian $\Delta^{(\kappa)}$, which we state as the next theorem.

Theorem 3.21. *Let $K_t^{(\kappa)}$ denote the heat kernel at a point $Z = k_0\, i \exp(2R) \in \mathbb{H}_n$ corresponding to the Siegel–Maaß Laplacian $\Delta^{(\kappa)}$ of weight κ on the Siegel upper half-space with $k_0 \in K_0$ and*

$$
R = \begin{pmatrix} r_1 & & 0 \\ & \ddots & \\ 0 & & r_n \end{pmatrix} \qquad (r_j \in \mathbb{R}_{\geq 0},\ 1 \leq j \leq n).
$$

Then, subject to the above conjecture, $K_t^{(\kappa)}$ is bounded above by

$$
K_t^{(\kappa)}(2R) \leq c_n \frac{\exp\left(-\sum_{j=1}^n j^2 t/4\right)}{t^{n^2+n/2}} \int\limits_{k \in K} \frac{\varepsilon(\varrho(r,k)) \exp\left(-\sum_{j=1}^n \left(\varrho_j(r,k)^2/t - \kappa|\varrho_j(r,k)|\right)\right)}{\delta(\varrho(r,k)) \prod_{j=1}^n \mathrm{ch}^\kappa(r_j)}\, \mathrm{d}\mu(k),
$$

where $\varrho(r,k)$ is the diagonal matrix $\varrho(r,k) = \left(\begin{smallmatrix} P(r,k) & 0 \\ 0 & -P(r,k) \end{smallmatrix}\right)$ with

$$
P(r,k) = \begin{pmatrix} \varrho_1(r,k) & & 0 \\ & \ddots & \\ 0 & & \varrho_n(r,k) \end{pmatrix} \qquad (\varrho_j(r,k) \in \mathbb{R},\ 1 \leq j \leq n)
$$

related to $r = \left(\begin{smallmatrix} R & 0 \\ 0 & -R \end{smallmatrix}\right)$ via the matrix equality $k \exp(r)\overline{k}^t = u \exp(\varrho)\overline{u}^t$ with $k \in K$ and $u \in U$. The functions ε and δ are given by

$$
\varepsilon(\varrho) = \prod_{1 \leq j \leq n} \varrho_j \prod_{1 \leq j < k \leq n} (\varrho_j + \varrho_k) \prod_{1 \leq j < k \leq n} (\varrho_j - \varrho_k),
$$

$$
\delta(\varrho) = \prod_{1 \leq j \leq n} \mathrm{sh}(\varrho_j) \prod_{1 \leq j < k \leq n} \mathrm{sh}\!\left(\frac{\varrho_j + \varrho_k}{2}\right) \prod_{1 \leq j < k \leq n} \mathrm{sh}\!\left(\frac{\varrho_j - \varrho_k}{2}\right),
$$

while c_n is a positive real constant depending only on n.

Proof. Follows immediately from Theorems 3.18, 3.17 and 3.20 $\qquad\qquad\square$

4 Sup-norm bounds on average

Let $K_t^{(\kappa)}(R(Z,W))\,(Z,W \in \mathbb{H}_n)$ denote the heat kernel on $\mathbb{H}_n$, where $R(Z,W)$ is the matrix

$$
R(Z,W) = \begin{pmatrix} r_1(Z,W) & & 0 \\ & \ddots & \\ 0 & & r_n(Z,W) \end{pmatrix} \qquad (r_j(Z,W) \in \mathbb{R}, 1 \leq j \leq n),
$$

with the entries $r_j(Z,W)\,(1 \leq j \leq n)$ of $R(Z,W)$ related to the eigenvalues $\rho_j(Z,W)\,(1 \leq j \leq n)$ of the cross-ratio matrix (see subsection 2.1)

$$
\rho(Z,W) = (Z-W)(\overline{Z}-W)^{-1}(\overline{Z}-\overline{W})(Z-\overline{W})^{-1} \quad (Z,W \in \mathbb{H}_n)
$$

by the relation

$$
\exp(2r_j(Z,W)) = \frac{1+\sqrt{\rho_j(Z,W)}}{1-\sqrt{\rho_j(Z,W)}} \qquad (1 \leq j \leq n).
$$

Then the heat kernel $K_t^{(\kappa,\Gamma)}$ on $\Gamma\backslash\mathbb{H}_n$ is given by the Γ-periodization

$$
K_t^{(\kappa,\Gamma)}(Z,W) := \sum_{\gamma \in \Gamma} \det\left(\frac{Z - \gamma\overline{W}}{\gamma W - \overline{Z}}\right)^{\kappa/2} \det\left(\frac{C\overline{W}+D}{CW+D}\right)^{\kappa/2} K_t^{(\kappa)}(2R(Z,\gamma W)).
$$

We write $K_t^{(\kappa,\Gamma)}(Z) := K_t^{(\kappa,\Gamma)}(Z,Z)$ and $R^\gamma(Z) := R(Z,\gamma Z)$ with entries

$$R^\gamma(Z) = \begin{pmatrix} r_1^\gamma(Z) & & 0 \\ & \ddots & \\ 0 & & r_n^\gamma(Z) \end{pmatrix} \quad (r_j^\gamma(Z) \in \mathbb{R}, 1 \le j \le n).$$

Since $\Delta^{(\kappa)}$ is symmetric, it extends to an essentially self-adjoint linear operator acting on a dense subspace of $\mathcal{H}_\kappa^n(\Gamma)$. Therefore the heat kernel $K_t^{(\kappa,\Gamma)}(Z,W)$ has a spectral decomposition

$$K_t^{(\kappa,\Gamma)}(Z,W) = \sum_{j=1}^\infty \exp(-\lambda_j t) \varphi_{\lambda_j}(Z) \overline{\varphi}_{\lambda_j}(W)$$
$$+ \sum_{\mathcal{P} \in \mathcal{C}} c_\mathcal{P} \int_{\lambda \in \mathfrak{a}_\mathcal{P}^\vee} \exp(-(\langle \rho_\mathcal{P}, \rho_\mathcal{P} \rangle + \langle \lambda, \lambda \rangle)t)\, E_\mathcal{P}(Z, \rho_\mathcal{P} + i\lambda)\, \overline{E}_\mathcal{P}(W, \rho_\mathcal{P} + i\lambda)\, \mathrm{d}\lambda \quad (4.1)$$

converging absolutely and uniformly on compacta for $t > 0$. The discrete part of the spectrum given by the first sum runs over the eigenvalues λ_j of the Siegel–Maaß Laplacian $\Delta^{(\kappa)}$ with eigenfunctions φ_{λ_j}. The continuous part of the spectrum given by the second sum runs over the set $\mathcal{C}$ of inequivalent chains of rational boundary components of M with $c_\mathcal{P}$ denoting a positive constant depending on the cusp $\mathcal{P} \in \mathcal{C}$, $\mathfrak{a}_\mathcal{P}$ the Lie algebra of the diagonal component $A_\mathcal{P}$ of $\mathcal{P}$, $\rho_\mathcal{P}$ the half-sum of positive roots with multiplicity in $\mathfrak{a}_\mathcal{P}$ and $E_\mathcal{P}$ the Eisenstein series attached to the cusp $\mathcal{P}$. Setting $Z = W$ in equation (4.1), we obtain

$$K_t^{(\kappa,\Gamma)}(Z) = \sum_{j=1}^\infty \exp(-\lambda_j t)\, |\varphi_{\lambda_j}(Z)|^2 + \sum_{\mathcal{P} \in \mathcal{C}} c_\mathcal{P} \int_{\lambda \in \mathfrak{a}_\mathcal{P}^\vee} \exp(-(\langle \rho_\mathcal{P}, \rho_\mathcal{P} \rangle + \langle \lambda, \lambda \rangle)t)\, |E_\mathcal{P}(Z, \rho_\mathcal{P} + i\lambda)|^2\, \mathrm{d}\lambda$$

Now, let $\kappa \ge n+1$ and multiply both sides of the above equation by $\exp((n\kappa/4)((n+1) - \kappa)t)$. Then

$$\frac{n\kappa}{4}((n+1) - \kappa) - \langle \rho_F, \rho_F \rangle - \langle \lambda, \lambda \rangle < 0.$$

Also, since $\lambda_j \ge (n\kappa/4)((n+1) - \kappa)$ by Theorem 2.26, we have

$$\frac{n\kappa}{4}((n+1) - \kappa) - \lambda_j \le 0.$$

Therefore, on taking limit $t \to \infty$ on both sides of the above equation, on the right-hand side only the φ_{λ_j}'s corresponding to $\lambda_j = (n\kappa/4)((n+1) - \kappa)$ survive. By Theorem 2.26, these are of the form $\varphi_{\lambda_j}(Z) = \det(Y)^{\kappa/2} f_j(Z)$. Therefore, we have

$$\lim_{t \to \infty} \exp\big(-\frac{n\kappa}{4}(\kappa - (n+1))\,t\big)\, K_t^{(\kappa,\Gamma)}(Z) = \sum_{j=1}^d (\det Y)^\kappa |f_j(Z)|^2 \quad (\kappa > (n+1)), \quad (4.2)$$

where $d = \dim(\mathcal{S}_\kappa^n(\Gamma))$ and $\{f_j\}_{1 \le j \le d}$ is an orthonormal basis of $\mathcal{S}_\kappa^n(\Gamma)$ with respect to the Petersson inner product. We denote

$$S_\kappa^\Gamma(Z) := \sum_{j=1}^d \det(Y)^\kappa |f_j(Z)|^2 \quad (Z \in \mathbb{H}_n).$$

Thus, we have

$$S_\kappa^\Gamma(Z) = \lim_{t \to \infty} \exp\big(-\frac{n\kappa}{4}(\kappa - (n+1))\,t\big)\, K_t^{(\kappa,\Gamma)}(Z). \quad (4.3)$$

Since the function $\exp(-n\kappa(\kappa - (n+1))\,t/4)K_t^{(\kappa,\Gamma)}(Z)$ is monotonically decreasing for any $t > 0$, we also have

$$S_\kappa^\Gamma(Z) \le \exp\big(-\frac{n\kappa}{4}(\kappa - (n+1))\,t\big)\, K_t^{(\kappa,\Gamma)}(Z).$$

Further, as

$$\left| \det\left(\frac{Z - \gamma\overline{Z}}{\gamma Z - \overline{Z}}\right)^{\kappa/2} \det\left(\frac{C\overline{Z} + D}{CZ + D}\right)^{\kappa/2} \right| = 1,$$

this also implies that for any $t > 0$ and $Z \in \mathbb{H}_n$, we have

$$S_\kappa^\Gamma(Z) \leq \exp\left(-\frac{n\kappa}{4}(\kappa - (n+1))\,t\right) \sum_{\gamma \in \Gamma} K_t^{(\kappa)}(2R^\gamma(Z)). \tag{4.4}$$

4.1 Sup-norm bounds in the cocompact setting

Note that, to make the calculations less cumbersome, we continue clubbing all positive real constants depending only on n under the generic symbol c_n.

Lemma 4.1. *Let G denote the complex symplectic group $\mathrm{Sp}_n(\mathbb{C})$, $K \subsetneq G$ denote the complex orthogonal group $K = \{k \in G \mid kk^t = \mathbb{1}_{2n}\}$ and $X = G/U$ denote the symmetric space $X = \{x = g\overline{g}^t \mid g \in G\}$. Then the invariant volume form $\mathrm{d}\mu(x)$ on X in the coordinates $x = k\exp(r)\overline{k}^t$ ($x \in X$), where r is given by the diagonal matrix $r = \left(\begin{smallmatrix} R & 0 \\ 0 & -R \end{smallmatrix}\right)$ with*

$$R = \begin{pmatrix} r_1 & & 0 \\ & \ddots & \\ 0 & & r_n \end{pmatrix} \qquad (r_j \in \mathbb{R},\ 1 \leq j \leq n),$$

is given by

$$\mathrm{d}\mu(x) = c_n\,|\delta(2r)| \bigwedge_{j=1}^n \mathrm{d}r_j \wedge \mathrm{d}\mu(k),$$

where $\mathrm{d}\mu(k)$ denotes the Haar measure on K, $\delta(r)$ denotes the function

$$\delta(r) = \prod_{1 \leq j \leq n} \mathrm{sh}(r_j) \prod_{1 \leq j < k \leq n} \mathrm{sh}\left(\frac{r_j + r_k}{2}\right) \prod_{1 \leq j < k \leq n} \mathrm{sh}\left(\frac{r_j - r_k}{2}\right)$$

on R and c_n is a constant depending only on n.

Proof. The tangent space of $X = G/U$ at identity is given by the space $\mathfrak{p}$ of real dimension $n(2n+1)$. Therefore, to calculate the invariant volume form $\mathrm{d}\mu(x)$ at $x \in X = G/U$, we first calculate the invariant matrix differential form $x^{-1}\,\mathrm{d}x \in \mathfrak{p}^\vee$. Then, for a choice of dual basis $e_1, \ldots, e_j, \ldots, e_{n(2n+1)}$ of $\mathfrak{p}^\vee$, we have

$$x^{-1}\,\mathrm{d}x = \omega_1(x)e_1 + \ldots + \omega_j(x)e_j + \ldots + \omega_{n(2n+1)}(x)e_{n(2n+1)},$$

where each $\omega_j(x)\,(1 \leq j \leq n(2n+1))$ is a real 1-form. The volume form $\mathrm{d}\mu(x)$, denoted by $[x^{-1}\,\mathrm{d}x]$ is then obtained by taking the wedge product

$$[x^{-1}\,\mathrm{d}x] = \omega_1(x) \wedge \cdots \omega_j(x) \cdots + \omega_{n(2n+1)}(x).$$

From $x = k\exp(r)\overline{k}^t$ ($x \in X$), one obtains

$$\begin{aligned} x^{-1}\,\mathrm{d}x &= \overline{k}e^{-r}k^t\,\mathrm{d}ke^r\overline{k}^t + \overline{k}\,\mathrm{d}r\overline{k}^t + \overline{k}\,\mathrm{d}\overline{k}^t \\ &= \overline{k}e^{-r/2}\big(e^{-r/2}(k^t\,\mathrm{d}k)e^{r/2} + e^{r/2}(\mathrm{d}r)e^{-r/2} + e^{r/2}(\mathrm{d}\overline{k}^t\overline{k})e^{-r/2}\big)e^{r/2}\overline{k}^t. \end{aligned}$$

Then taking volume form, denoted by the parentheses $[\,\cdot\,]$, on both sides, we have

$$\mathrm{d}\mu(x) = [x^{-1}\,\mathrm{d}x] = [e^{-r/2}(k^t\,\mathrm{d}k)e^{r/2} + e^{r/2}(\mathrm{d}r)e^{-r/2} + e^{r/2}(\mathrm{d}\overline{k}^t\overline{k})e^{-r/2}]. \tag{4.5}$$

Now, as the invariant differential form $k^t \, \mathrm{d}k$ has the structure of the elements of the complex orthogonal Lie algebra $\mathfrak{k}$, we take

$$k^t \, \mathrm{d}k = \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \qquad (A, B \in \mathbb{C}^{n \times n}, \ A = -A^t, \ B = B^t).$$

Then the form $\mathrm{d}\overline{k}^t \overline{k} = \overline{k^t \, \mathrm{d}k}^t$ is given by

$$\mathrm{d}\overline{k}^t \overline{k} = \begin{pmatrix} \overline{A}^t & -\overline{B}^t \\ \overline{B}^t & \overline{A}^t \end{pmatrix} = \begin{pmatrix} -\overline{A} & -\overline{B} \\ \overline{B} & -\overline{A} \end{pmatrix}.$$

Then writing the right-hand side of equation (4.5) in block decomposed form, we have

$$\begin{aligned}
\mathrm{d}\mu(x) = &\left[\begin{pmatrix} e^{-R/2} & 0 \\ 0 & e^{R/2} \end{pmatrix} \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \begin{pmatrix} e^{R/2} & 0 \\ 0 & e^{-R/2} \end{pmatrix} \right. \\
&+ \left. \begin{pmatrix} e^{R/2} & 0 \\ 0 & e^{-R/2} \end{pmatrix} \begin{pmatrix} -\overline{A} & -\overline{B} \\ \overline{B} & -\overline{A} \end{pmatrix} \begin{pmatrix} e^{-R/2} & 0 \\ 0 & e^{R/2} \end{pmatrix} + \begin{pmatrix} \mathrm{d}R & 0 \\ 0 & -\mathrm{d}R \end{pmatrix} \right] \\
= &\left[\begin{pmatrix} e^{-R/2} A e^{R/2} - e^{R/2} \overline{A} e^{-R/2} & e^{-R/2} B e^{-R/2} - e^{R/2} \overline{B} e^{R/2} \\ -e^{R/2} B e^{R/2} + e^{-R/2} \overline{B} e^{-R/2} & e^{R/2} A e^{-R/2} - e^{-R/2} \overline{A} e^{R/2} \end{pmatrix} + \begin{pmatrix} \mathrm{d}R & 0 \\ 0 & -\mathrm{d}R \end{pmatrix} \right].
\end{aligned}$$

Now, writing the matrices A and B as

$$\begin{aligned}
A &= \big(\alpha_{j,k} = \xi_{j,k} + i\eta_{j,k} \big)_{1 \le j,k \le n} & \big(\alpha_{j,j} = 0, \ \alpha_{k,j} = -\alpha_{j,k} \ (1 \le j < k \le n) \big), \\
B &= \big(\beta_{j,k} = \omega_{j,k} + i\tau_{j,k} \big)_{1 \le j,k \le n} & \big(\beta_{k,j} = \beta_{j,k} \ (1 \le j \le k \le n) \big),
\end{aligned}$$

where $\xi_{j,k}, \eta_{j,k}, \omega_{j,k}, \tau_{j,k}$ are real 1-forms, one obtains

$$\begin{aligned}
\big(e^{-R/2} A e^{R/2} - e^{R/2} \overline{A} e^{-R/2} \big)_{j,k} &= e^{(r_k - r_j)/2} (\xi_{j,k} + i\eta_{j,k}) - e^{(r_j - r_k)/2} (\xi_{j,k} - i\eta_{j,k}) \\
&= -2 \operatorname{sh}\!\Big(\frac{r_j - r_k}{2} \Big) \xi_{j,k} + 2i \operatorname{ch}\!\Big(\frac{r_j - r_k}{2} \Big) \eta_{j,k} \ (1 \le j < k \le n)
\end{aligned}$$

and similarly

$$\begin{aligned}
\big(e^{R/2} A e^{-R/2} - e^{-R/2} \overline{A} e^{R/2} \big)_{j,k} &= 2 \operatorname{sh}\!\Big(\frac{r_j - r_k}{2} \Big) \xi_{j,k} + 2i \operatorname{ch}\!\Big(\frac{r_j - r_k}{2} \Big) \eta_{j,k} \ (1 \le j < k \le n), \\
\big(e^{-R/2} B e^{-R/2} - e^{R/2} \overline{B} e^{R/2} \big)_{j,k} &= -2 \operatorname{sh}\!\Big(\frac{r_j + r_k}{2} \Big) \omega_{j,k} + 2i \operatorname{ch}\!\Big(\frac{r_j + r_k}{2} \Big) \tau_{j,k} \ (1 \le j \le k \le n), \\
\big(e^{R/2} B e^{R/2} - e^{-R/2} \overline{B} e^{-R/2} \big)_{j,k} &= 2 \operatorname{sh}\!\Big(\frac{r_j + r_k}{2} \Big) \omega_{j,k} + 2i \operatorname{ch}\!\Big(\frac{r_j + r_k}{2} \Big) \tau_{j,k} \ (1 \le j \le k \le n).
\end{aligned}$$

Now taking wedge product of the above entries, it easily follows that

$$\mathrm{d}\mu(x) = c_n \, |\delta(2r)| \bigwedge_{j=1}^{n} \mathrm{d}r_j \bigwedge_{1 \le j < k \le n} (\xi_{j,k} \wedge \eta_{j,k}) \bigwedge_{1 \le j \le k \le n} (\omega_{j,k} \wedge \tau_{j,k}),$$

whence, identifying

$$\mathrm{d}\mu(k) = \bigwedge_{1 \le j < k \le n} (\xi_{j,k} \wedge \eta_{j,k}) \bigwedge_{1 \le j \le k \le n} (\omega_{j,k} \wedge \tau_{j,k})$$

we have the result stated in the lemma. $\qquad\square$

Lemma 4.2. *Let G denote the complex symplectic group $\mathrm{Sp}_n(\mathbb{C})$, $U \subsetneq G$ denote the symplectic unitary group $U = \{u \in G \mid u\overline{u}^t = \mathbb{1}_{2n}\}$ and $X = G/U$ denote the symmetric space $X = \{x = g\overline{g}^t \mid g \in G\}$. Then the invariant volume form $\mathrm{d}\mu(x)$ on X in the coordinates $x = u \exp(\varrho)\overline{u}^t$ $(x \in X)$, where ϱ is given by the diagonal matrix $\varrho = \big(\begin{smallmatrix} P & 0 \\ 0 & -P \end{smallmatrix} \big)$ with*

$$P = \begin{pmatrix} \varrho_1 & & 0 \\ & \ddots & \\ 0 & & \varrho_n \end{pmatrix} \qquad (\varrho_j \in \mathbb{R}, \ 1 \le j \le n),$$

is given by

$$\mathrm{d}\mu(x) = c_n\,\delta(\varrho)^2 \bigwedge_{j=1}^{n} \mathrm{d}\varrho_j \wedge \mathrm{d}\mu(u),$$

where $\mathrm{d}\mu(u)$ denotes the Haar measure on U, $\delta(\varrho)$ denotes the function

$$\delta(\varrho) = \prod_{1 \le j \le n} \mathrm{sh}(\varrho_j) \prod_{1 \le j < k \le n} \mathrm{sh}\!\left(\frac{\varrho_j + \varrho_k}{2}\right) \prod_{1 \le j < k \le n} \mathrm{sh}\!\left(\frac{\varrho_j - \varrho_k}{2}\right)$$

on P and c_n is a constant depending only on n.

Proof. Proceeding as in the proof of Lemma 4.1, from $x = u\exp(\varrho)\overline{u}^t$, one obtains

$$x^{-1}\,\mathrm{d}x = ue^{-\varrho}\overline{u}^t\,\mathrm{d}ue^{\varrho}\overline{u}^t + u\,\mathrm{d}\varrho\overline{u}^t + u\,\mathrm{d}\overline{u}^t$$
$$= ue^{-\varrho/2}\big(e^{-\varrho/2}(\overline{u}^t\,\mathrm{d}u)e^{\varrho/2} + e^{\varrho/2}(\mathrm{d}\rho)e^{-\varrho/2} + e^{\varrho/2}(\mathrm{d}\overline{u}^t u)e^{-\varrho/2}\big)e^{\varrho/2}\overline{u}^t.$$

Then taking volume form, denoted by the parentheses $[\,\cdot\,]$, on both sides, we have

$$\mathrm{d}\mu(x) = [x^{-1}\,\mathrm{d}x] = [e^{-\varrho/2}(\overline{u}^t\,\mathrm{d}u)e^{\varrho/2} + e^{\varrho/2}(\mathrm{d}\rho)e^{-\varrho/2} + e^{\varrho/2}(\mathrm{d}\overline{u}^t u)e^{-\varrho/2}]. \qquad (4.6)$$

Now, as the invariant differential form $\overline{u}^t\,\mathrm{d}u$ has the structure of the elements of the unitary symplectic Lie algebra $\mathfrak{u}$, we take

$$\overline{u}^t\,\mathrm{d}u = \begin{pmatrix} A & B \\ -\overline{B} & \overline{A} \end{pmatrix} \qquad (A, B \in \mathbb{C}^{n \times n},\ \overline{A}^t = -A,\ B = B^t).$$

Also, from $\overline{u}^t u = \mathbb{1}_{2n}$ it follows that that $\mathrm{d}\overline{u}^t u = -\overline{u}^t\,\mathrm{d}u$. Then writing the right-hand side of equation (4.6) in block decomposed form, we have

$$\mathrm{d}\mu(x) = \Bigg[\begin{pmatrix} e^{-P/2} & 0 \\ 0 & e^{P/2} \end{pmatrix} \begin{pmatrix} A & B \\ -\overline{B} & \overline{A} \end{pmatrix} \begin{pmatrix} e^{P/2} & 0 \\ 0 & e^{-P/2} \end{pmatrix}$$
$$- \begin{pmatrix} e^{P/2} & 0 \\ 0 & e^{-P/2} \end{pmatrix} \begin{pmatrix} A & B \\ -\overline{B} & \overline{A} \end{pmatrix} \begin{pmatrix} e^{-P/2} & 0 \\ 0 & e^{P/2} \end{pmatrix} + \begin{pmatrix} \mathrm{d}P & 0 \\ 0 & -\mathrm{d}P \end{pmatrix} \Bigg]$$
$$= \Bigg[\begin{pmatrix} e^{-P/2}Ae^{P/2} - e^{P/2}Ae^{-P/2} & e^{-P/2}Be^{-P/2} - e^{P/2}Be^{P/2} \\ -e^{P/2}\overline{B}e^{P/2} + e^{-P/2}\overline{B}e^{-P/2} & e^{P/2}\overline{A}e^{-P/2} - e^{-P/2}\overline{A}e^{P/2} \end{pmatrix} + \begin{pmatrix} \mathrm{d}P & 0 \\ 0 & -\mathrm{d}P \end{pmatrix} \Bigg].$$

Now, writing the matrices A and B as

$$A = \big(\alpha_{j,k}\big)_{1 \le j,k \le n} \qquad \big(\mathrm{Re}(\alpha_{j,j}) = 0,\ \alpha_{k,j} = -\overline{\alpha_{j,k}}\ (1 \le j < k \le n)\big),$$
$$B = \big(\beta_{j,k}\big)_{1 \le j,k \le n} \qquad \big(\beta_{k,j} = \beta_{j,k}\ (1 \le j \le k \le n)\big),$$

where $\alpha_{j,k}$, $\beta_{j,k}$ are complex 1-forms, one obtains

$$\big(e^{-P/2}Ae^{P/2} - e^{P/2}Ae^{-P/2}\big)_{j,k} = -2\,\mathrm{sh}\!\left(\frac{\varrho_j - \varrho_k}{2}\right)\alpha_{j,k} \qquad (1 \le j < k \le n),$$
$$\big(e^{P/2}\overline{A}e^{-P/2} - e^{-P/2}\overline{A}e^{P/2}\big)_{j,k} = 2\,\mathrm{sh}\!\left(\frac{\varrho_j - \varrho_k}{2}\right)\overline{\alpha}_{j,k} \qquad (1 \le j < k \le n),$$
$$\big(e^{-P/2}Be^{-P/2} - e^{P/2}Be^{P/2}\big)_{j,k} = -2\,\mathrm{sh}\!\left(\frac{\varrho_j + \varrho_k}{2}\right)\beta_{j,k} \qquad (1 \le j \le k \le n),$$
$$\big(e^{P/2}\overline{B}e^{P/2} + e^{-P/2}\overline{B}e^{-P/2}\big)_{j,k} = 2\,\mathrm{sh}\!\left(\frac{\varrho_j + \varrho_k}{2}\right)\overline{\beta}_{j,k} \qquad (1 \le j \le k \le n).$$

Now taking wedge product of the above entries, it easily follows that

$$\mathrm{d}\mu(x) = c_n\,\delta(\varrho)^2 \bigwedge_{j=1}^{n} \mathrm{d}\varrho_j \bigwedge_{1 \le j < k \le n} (\alpha_{j,k} \wedge \overline{\alpha}_{j,k}) \bigwedge_{1 \le j \le k \le n} (\beta_{j,k} \wedge \overline{\beta}_{j,k}),$$

whence, identifying

$$\mathrm{d}\mu(u) = \bigwedge_{1 \le j < k \le n} (\alpha_{j,k} \wedge \overline{\alpha}_{j,k}) \bigwedge_{1 \le j \le k \le n} (\beta_{j,k} \wedge \overline{\beta}_{j,k})$$

we have the result stated in the lemma. $\qquad \square$

Theorem 4.3. *For any arithmetic subgroup* $\Gamma \subsetneqq \mathrm{Sp}_n(\mathbb{R})$ *such that* $M := \Gamma \backslash \mathbb{H}_n$ *is compact, we have*

$$\sup_{Z \in \mathbb{H}_n} S_\kappa^\Gamma(Z) \le c_{n,\Gamma}\, \kappa^{n(n+1)/2} \qquad (\kappa \ge n+1),$$

where $c_{n,\Gamma}$ *is a positive real constant depending only on* n *and* Γ.

Proof. For $Z, W \in \mathbb{H}_n$, let $R(Z,W)$ denote the matrix

$$R(Z,W) = \begin{pmatrix} r_1(Z,W) & & 0 \\ & \ddots & \\ 0 & & r_n(Z,W) \end{pmatrix} \quad (r_j(Z,W) \in \mathbb{R}, 1 \le j \le n),$$

with the entries $r_j(Z,W)\,(1 \le j \le n)$ of $R(Z,W)$ related to the eigenvalues $\rho_j(Z,W)\,(1 \le j \le n)$ of the cross-ratio matrix

$$\rho(Z,W) = (Z - W)(\overline{Z} - W)^{-1}(\overline{Z} - \overline{W})(Z - \overline{W})^{-1} \quad (Z, W \in \mathbb{H}_n)$$

by the relation

$$\exp(2r_j(Z,W)) = \frac{1 + \sqrt{\rho_j(Z,W)}}{1 - \sqrt{\rho_j(Z,W)}} \qquad (1 \le j \le n).$$

Let $R(Z)$ denote the matrix $R(Z, i\mathbb{1}_n)$ with corresponding diagonal entries $r_j(Z)\,(1 \le j \le n)$ and for $\gamma \in \Gamma$, let $R^\gamma(Z)$ denote the matrix $R(Z, \gamma Z)$ with corresponding diagonal entries $r_j^\gamma(Z)\,(1 \le j \le n)$.

Now, since $M = \Gamma \backslash \mathbb{H}_n$ is compact, there are only finitely many elements $\gamma \in \Gamma$, namely, the torsion elements of Γ, for which the point γZ can get arbitrarily close to Z. Then, denoting the set of torsion elements of Γ by Γ_T, there is a positive real constant $c_{n,\Gamma}$, such that $R^\gamma(Z) \ge c_{n,\Gamma} \mathbb{1}_n$ for all $\gamma \in \Gamma \backslash \Gamma_T$ and $Z \in \mathbb{H}_n$. Therefore, given n positive real numbers $r_j\,(1 \le j \le n)$, we have

$$\#\{\gamma \in \Gamma \mid r_j^\gamma(Z) \le r_j,\, 1 \le j \le n\} \le c_{n,\Gamma} \,\mathrm{vol}_n(\{Z \in \mathbb{H}_n \mid r_j(Z) \le r_j\ 1 \le j \le n\}),$$

for some positive real positive constant $c_{n,\Gamma}$ depending only on n and Γ. The dependence on Γ here is given by the maximal order of the torsion elements of Γ.

As the volume form on $\mathbb{H}_n$ in polar coordinates is given by

$$\mathrm{d}\mu_n(k_0 \exp(2R)i) = |\delta(2r)| \bigwedge_{j=1}^n \mathrm{d}r_j \wedge \mathrm{d}\mu(k_0)$$

with $R = \mathrm{diag}(r_1, r_2, \ldots, r_n)$, $r = \left(\begin{smallmatrix} R & 0 \\ 0 & -R \end{smallmatrix}\right)$ and $k_0 \in K_0 = \mathrm{Sp}_n(\mathbb{R}) \cap \mathrm{O}(2n, \mathbb{R})$, we have

$$\mathrm{dvol}_n(\{Z \in \mathbb{H}_n \mid r_j(Z) \le r_j,\, 1 \le j \le n\}) = |\delta(2r)| \bigwedge_{j=1}^n \mathrm{d}r_j.$$

Therefore, as the heat kernel $K_t^{(\kappa)}(2R)$ is non-negative, continuous, and monotonically decreasing in each $r_j\,(1 \le j \le n)$, we have

$$\sum_{\gamma \in \Gamma} K_t^{(\kappa)}(R^\gamma(Z)) \le c_{n,\Gamma} \int_{r_1=0}^\infty \cdots \int_{r_n=0}^\infty K_t^{(\kappa)}(2R)\, |\delta(2r)| \bigwedge_{j=1}^n \mathrm{d}r_j.$$

Hence, from equation (4.4) and Theorem 3.21, we have

$$S_\kappa^\Gamma(Z) \le c_{n,\Gamma}\, I_n(\kappa, t), \tag{4.7}$$

where the function $I_n(\kappa, t)$ is given by the integral

$$I_n(\kappa, t) := \frac{\exp\left(\left(-n\kappa(\kappa - (n+1)) - \sum_{j=1}^n j^2\right)t/4\right)}{t^{n^2+n/2}}$$

$$\int_{r_1=0}^\infty \cdots \int_{r_n=0}^\infty \int_{k \in K} \frac{\varepsilon(\varrho(r,k)) \exp\left(-\sum_{j=1}^n \left(\varrho_j(r,k)^2/t - \kappa|\varrho_j(r,k)|\right)\right)}{\delta(\varrho(r,k)) \prod_{j=1}^n \mathrm{ch}^\kappa(r_j)} |\delta(2r)| \bigwedge_{j=1}^n \mathrm{d}r_j \bigwedge \mathrm{d}\mu(k),$$

Now, using Lemmas 4.1 and 4.2, we switch the above integral from over r, k $(k \in K)$ coordinates on $X = G/U$ to ϱ, u $(u \in U)$ coordinates on X. As we have

$$|\delta(2r)| \bigwedge_{j=1}^{n} \mathrm{d}r_j \wedge \mathrm{d}\mu(k) = c_n \, \delta(\varrho)^2 \bigwedge_{j=1}^{n} \mathrm{d}\varrho_j \wedge \mathrm{d}\mu(u),$$

with this change of variables, the above integral becomes

$$I_n(\kappa, t) = c_n \frac{\exp\left(\left(-n\kappa(\kappa - (n+1)) - \sum_{j=1}^{n} j^2\right) t/4\right)}{t^{n^2 + n/2}}$$

$$\int_{\varrho_1 = -\infty}^{\infty} \cdots \int_{\varrho_n = -\infty}^{\infty} \int_{u \in U} \frac{\varepsilon(\varrho) \exp\left(-\sum_{j=1}^{n} \left(\varrho_j^2/t - \kappa|\varrho_j|\right)\right)}{\prod_{j=1}^{n} \mathrm{ch}^\kappa(r_j)} \delta(\varrho) \bigwedge_{j=1}^{n} \mathrm{d}\varrho_j \bigwedge \mathrm{d}\mu(u)$$

$$= c_n \frac{\exp\left(\left(-n\kappa(\kappa - (n+1)) - \sum_{j=1}^{n} j^2\right) t/4\right)}{t^{n^2 + n/2}}$$

$$\int_{\varrho_1 = -\infty}^{\infty} \cdots \int_{\varrho_n = -\infty}^{\infty} \varepsilon(\varrho)\delta(\varrho) \exp\left(-\sum_{j=1}^{n} \left(\varrho_j^2/t - \kappa|\varrho_j|\right)\right) J_n(\varrho, \kappa) \bigwedge_{j=1}^{n} \mathrm{d}\varrho_j,$$

where $J_n(\varrho, \kappa)$ is the integral given by

$$J_n(\varrho, \kappa) := \int_{u \in U} \frac{\mathrm{d}\mu(u)}{\prod_{j=1}^{n} \mathrm{ch}^\kappa(r_j)}.$$

Now, as $1/\mathrm{ch}(r_j) \le 2\exp(-r_j)$ $(1 \le j \le n)$ and $\kappa \ge n+1$ we have

$$J_n(\varrho, \kappa) \le c_n \int_{u \in U} \exp\left(-\kappa \sum_{j=1}^{n} r_j\right) \mathrm{d}\mu(u) \le c_n \int_{u \in U} \exp\left(-(n+1) \sum_{j=1}^{n} r_j\right) \mathrm{d}\mu(u).$$

Then, as $r_j \in \mathbb{R}_{\ge 0}$ $(1 \le j \le n)$, from Lemma 3.4, we have

$$\phi_0(\exp(r)) \ge \exp(-\rho(r)) = \exp(-nr_1 - (n-1)r_2 - \ldots - r_n) \ge \exp\left(-(n+1) \sum_{j=1}^{n} r_j\right),$$

where ϕ_0 is the real spherical function on $\mathbb{H}_n$ corresponding to $\lambda = 0 \in \mathfrak{a}^\vee$. Then, we have

$$J_n(\varrho, \kappa) \le c_n \int_{u \in U} \phi_0(\exp(r)) \, \mathrm{d}\mu(u) = c_n \, \Phi_0(\exp(\varrho)),$$

where $\Phi_0(\exp(\varrho))$ is the complex spherical function on $X = G/U$ corresponding to $\lambda = 0$. Now, from Proposition 3.14, we have

$$J_n(\varrho, \kappa) \le c_n \frac{\varepsilon(\varrho)}{\delta(2\varrho)}.$$

Putting $J_n(\varrho, \kappa)$ back in $I_n(\kappa, t)$, we have

$$I_n(\kappa, t) \le c_n \frac{\exp\left(\left(-n\kappa(\kappa - (n+1)) - \sum_{j=1}^{n} j^2\right) t/4\right)}{t^{n^2 + n/2}}$$

$$\int_{\varrho_1 = -\infty}^{\infty} \cdots \int_{\varrho_n = -\infty}^{\infty} \frac{\varepsilon(\varrho)^2 \exp\left(-\sum_{j=1}^{n} \left(\varrho_j^2/t - \kappa|\varrho_j|\right)\right)}{\nu(\varrho)} \bigwedge_{j=1}^{n} \mathrm{d}\varrho_j,$$

where $\nu(\varrho)$ is the function given by

$$\nu(\varrho) = \prod_{1 \le j \le n} \mathrm{ch}(\varrho_j) \prod_{1 \le j < k \le n} \mathrm{ch}\left(\frac{\varrho_j + \varrho_k}{2}\right) \prod_{1 \le j < k \le n} \mathrm{ch}\left(\frac{\varrho_j - \varrho_k}{2}\right).$$

Now, since the integrand in the above integral is an even function of each ϱ_j ($1 \le j \le n$), we can integrate each ϱ_j in the limit $\varrho_j \in [0, \infty]$, thereby giving

$$I_n(\kappa, t) \le c_n \frac{\exp\big(\big(-n\kappa(\kappa - (n+1)) - \sum_{j=1}^n j^2\big)t/4\big)}{t^{n^2+n/2}}$$
$$\int_{\varrho_1=0}^{\infty} \cdots \int_{\varrho_n=0}^{\infty} \frac{\varepsilon(\varrho)^2 \exp\big(-\sum_{j=1}^n \big(\varrho_j^2/t - \kappa\varrho_j\big)\big)}{\nu(\varrho)} \bigwedge_{j=1}^n \mathrm{d}\varrho_j$$

Now, as $1/\nu(\varrho) \le c_n \exp(-n\varrho_1 - (n-1)\varrho_2 - \ldots - \varrho_n)$ and $n\kappa(\kappa - (n+1)) + \sum_{j=1}^n j^2 = (\kappa - n)^2 + \ldots + (\kappa - 1)^2$, we have

$$I_n(\kappa, t) \le c_n \int_{\varrho_1=0}^{\infty} \cdots \int_{\varrho_n=0}^{\infty} \frac{\varepsilon(\varrho)^2 \exp\big(-\sum_{j=1}^n (\varrho_j/\sqrt{t} - (\kappa - (n-j+1))\sqrt{t}/2)^2\big)}{t^{n^2+n/2}} \bigwedge_{j=1}^n \mathrm{d}\varrho_j$$

$$\le c_n \int_{\varrho_1=-\infty}^{\infty} \cdots \int_{\varrho_n=-\infty}^{\infty} \frac{\varepsilon(\varrho)^2 \exp\big(-\sum_{j=1}^n (\varrho_j/\sqrt{t} - (\kappa - (n-j+1))\sqrt{t}/2)^2\big)}{t^{n^2+n/2}} \bigwedge_{j=1}^n \mathrm{d}\varrho_j;$$

in the sequel we denote the latter integral by $H_n(\kappa, t)$. Then setting $\xi_j = \varrho_j/\sqrt{t} - (\kappa - (n-j+1))\sqrt{t}/2$, we have

$$\varrho_j = \xi_j\sqrt{t} + (\kappa - (n-j+1))t/2,$$

whence one obtains

$$\bigwedge_{j=1}^n \mathrm{d}\varrho_j = t^{n/2} \bigwedge_{j=1}^n \mathrm{d}\xi_j.$$

The quantity $\varepsilon(\varrho)^2$ now becomes

$$\varepsilon(\varrho)^2 = \prod_{1 \le j \le n} \varrho_j^2 \prod_{1 \le l < m \le n} (\varrho_l - \varrho_m)^2 \prod_{1 \le l < m \le n} (\varrho_l + \varrho_m)^2$$

$$= t^{n^2} \prod_{1 \le j \le n} \left(\xi_j + \frac{(\kappa - (n-j+1))\sqrt{t}}{2}\right)^2 \prod_{1 \le l < m \le n} \left((\xi_l - \xi_m) + \frac{l-m}{2}\sqrt{t}\right)^2 \times$$

$$\times \prod_{1 \le l < m \le n} \left((\xi_l + \xi_m) + \left(\kappa - \left(n - \frac{l+m}{2} + 1\right)\right)\sqrt{t}\right)^2,$$

which is a polynomial in $\xi = (\xi_1, \ldots, \xi_n)$, κ and t. Then, putting $\varepsilon(\varrho)^2$ back in **??**, we have

$$H_n(\kappa, t) = \int_{\xi_1=-\infty}^{\infty} \cdots \int_{\xi_n=-\infty}^{\infty} \exp\big(-\sum_{j=1}^n \xi_j^2\big) \prod_{1 \le l < m \le n} \left((\xi_l + \xi_m) + \left(\kappa - \left(n - \frac{l+m}{2} + 1\right)\right)\sqrt{t}\right)^2 \times$$

$$\times \prod_{1 \le j \le n} \left(\xi_j + \frac{(\kappa - (n-j+1))\sqrt{t}}{2}\right)^2 \prod_{1 \le l < m \le n} \left((\xi_l - \xi_m) + \frac{l-m}{2}\sqrt{t}\right)^2 \bigwedge_{j=1}^n \mathrm{d}\xi_j.$$

After evaluating integrals of the form

$$\int_{\xi=-\infty}^{\infty} \xi^m \exp(-\xi^2)\, \mathrm{d}\xi = \frac{1 + (-1)^m}{2} \Gamma\Big(\frac{m+1}{2}\Big) \qquad (m \in \mathbb{N}_{\ge 0}),$$

$H_n(\kappa, t)$ becomes a polynomial in κ and t. As $\kappa \ge n + 1 > 0$ and $t > 0$, for an upper bound, we need only consider the highest powers of κ and t in $H_n(\kappa, t)$. From the above formula, it follows that

$$H_n(\kappa, t) \le c_n\, \kappa^{n(n+1)}\, t^{n(n+1)/2} \mu(\sqrt{t}),$$

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where μ is a polynomial of order $n(n-1)$. Thus, from equations (4.7) and (**??**), one obtains

$$S_\kappa^\Gamma(Z) \le c_{n,\Gamma}\, \kappa^{n(n+1)}\, t^{n(n+1)/2} \mu(\sqrt{t}) \qquad (Z \in \mathbb{H}_n).$$

Now multiplying both sides of the above inequality by $\exp(-\kappa t)$ and integrating over $t \in [0,\infty]$, we have

$$\int\limits_{t=0}^{\infty} \exp(-\kappa t) S_\kappa^\Gamma(Z)\,\mathrm{d}t = \frac{S_\kappa^\Gamma(Z)}{\kappa} \le c_{n,\Gamma}\, \kappa^{n(n+1)} \int\limits_{t=0}^{\infty} \exp(-\kappa t)\, t^{n(n+1)/2} \mu(\sqrt{t})\,\mathrm{d}t \le c_{n,\Gamma}\, \frac{\kappa^{n(n+1)}}{\kappa^{n(n+1)/2+1}},$$

whence it easily follows that

$$S_\kappa^\Gamma(Z) \le c_{n,\Gamma}\, \kappa^{n(n+1)/2} \qquad (Z \in \mathbb{H}_n)$$

thereby proving the result stated in the theorem. $\qquad\square$

4.2 Sup-norm bounds in the cofinite setting

Theorem 4.4. *For any arithmetic subgroup $\Gamma \subsetneq \mathrm{Sp}_n(\mathbb{R})$ such that $M := \Gamma\backslash\mathbb{H}_n$ is of finite volume, we have*

$$\sup_{Z \in \mathbb{H}_n} S_\kappa^\Gamma(Z) \le c_n\, \kappa^{n(n+1)/2} \sum_{\gamma \in \Gamma} \frac{1}{\prod_{j=1}^n \mathrm{ch}^\kappa(r_j^\gamma(Z))} \qquad (\kappa \ge n+1),$$

where $r_j^\gamma(Z)$ denotes the diagonal entries of the diagonal matrix $R^\gamma(Z) = R(Z, \gamma Z)$ and c_n is a positive real constant depending only on n and Γ.

Proof. From equation (4.3), we have

$$S_\kappa^\Gamma(Z) = \lim_{t \to \infty} \exp\Big(-\frac{n\kappa}{4}(\kappa - (n+1))t \Big) \sum_{\gamma \in \Gamma} K_t^{(\kappa)}(2R^\gamma(Z)) \tag{4.8}$$

and from Theorem 3.21, we have

$$K_t^{(\kappa)}(2R^\gamma(Z)) \le \frac{I_n(\kappa, t, R^\gamma(Z))}{\prod_{j=1}^n \mathrm{ch}^\kappa(r_j^\gamma(Z))},$$

where $I_n(\kappa, t, R^\gamma(Z))$ is the integral given by

$$I_n(\kappa, t, R^\gamma(Z)) = c_n \frac{\exp\big(-\sum_{j=1}^n j^2 t/4 \big)}{t^{n^2+n/2}} \int\limits_{k \in K} \frac{\varepsilon(\varrho) \exp\big(-\sum_{j=1}^n \big(\varrho_j^2/t - \kappa|\varrho_j| \big) \big)}{\delta(\varrho)}\,\mathrm{d}\mu(k).$$

Here $\varrho = \varrho(r^\gamma(Z), k)$ is the diagonal matrix $\varrho(r^\gamma(Z), k) = \big(\begin{smallmatrix} P(r^\gamma(Z),k) & 0 \\ 0 & -P(r^\gamma(Z),k) \end{smallmatrix} \big)$ with

$$P(r^\gamma(Z), k) = \begin{pmatrix} \varrho_1(r^\gamma(Z), k) & & 0 \\ & \ddots & \\ 0 & & \varrho_n(r^\gamma(Z), k) \end{pmatrix} \qquad (\varrho_j(r^\gamma(Z), k) \in \mathbb{R},\ 1 \le j \le n)$$

related to

$$r^\gamma(Z) = \begin{pmatrix} R^\gamma(Z) & 0 \\ 0 & -R^\gamma(Z) \end{pmatrix}$$

via the matrix equality

$$k \exp(r^\gamma(Z)) \overline{k}^t = u \exp(\varrho) \overline{u}^t \qquad (k \in K,\ u \in U). \tag{4.9}$$

Since heat kernels decrease rapidly with increasing distance, the integral $I_n(\kappa, t, R^\gamma(Z))$ also decreases rapidly with increasing distance and hence we have

$$I_n(\kappa, t, R^\gamma(Z)) \le I_n(\kappa, t, 0_n). \tag{4.10}$$

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Then, for $R^\gamma(Z) = 0_n$, the matrix equality (4.9) becomes

$$k\overline{k}^t = u\exp(\varrho)\overline{u}^t \qquad (k \in K,\, u \in U). \tag{4.11}$$

For the eigendecomposition (4.11) of $k\overline{k}^t$, we next determine u and $\exp(\varrho)$ in terms of $k \in K$. For this, we use the matrix l from (3.25) and calculate

$$k\overline{k}^t = \begin{pmatrix} A & B \\ -B & A \end{pmatrix} = l^{-1}\begin{pmatrix} A+iB & 0 \\ 0 & A-iB \end{pmatrix} l, \tag{4.12}$$

where the matrix $h := A + iB$ is Hermitian, as $k\overline{k}^t$ is Hermitian; note that $A - iB = h^{-t}$. Since h is Hermitian, we have

$$h = vD\overline{v}^t,$$

where $v \in U_n$ and D is a real diagonal $(n \times n)$-matrix. Substituting this into (4.12) yields

$$k\overline{k}^t = l^{-1}\begin{pmatrix} v & 0 \\ 0 & \overline{v} \end{pmatrix}\begin{pmatrix} D & 0 \\ 0 & D^{-1} \end{pmatrix}\begin{pmatrix} \overline{v}^t & 0 \\ 0 & v^t \end{pmatrix} l.$$

In this way, the factors on the right-hand side of (4.11) become

$$u = l^{-1}\begin{pmatrix} v & 0 \\ 0 & \overline{v} \end{pmatrix} \quad \text{and} \quad \exp(\varrho) = \begin{pmatrix} D & 0 \\ 0 & D^{-1} \end{pmatrix}. \tag{4.13}$$

Note that the eigendecomposition (4.11) is unique only up to the ordering of the eigenvalues $\exp(\pm\varrho_j)$ ($1 \le j \le n$), i.e., we can always choose $u \in U$ in such a way so that $\varrho_j \in \mathbb{R}_{\ge 0}$ ($1 \le j \le n$). Therefore, without loss of generality, for the rest of the calculation, we assume $\varrho_j \in \mathbb{R}_{\ge 0}$ ($1 \le j \le n$).

Next, we determine the invariant volume form $d\mu(k)$ in terms of ϱ and v by proceeding as in the proof of Lemma 4.1. From $x = k\overline{k}^t = u\exp(\varrho)\overline{u}^t$, one obtains

$$dx = dk\,\overline{k}^t + k\,d\overline{k}^t = du\exp(\varrho)\overline{u}^t + u\exp(\varrho)\,d\varrho\,\overline{u}^t + u\exp(\varrho)\,d\overline{u}^t.$$

Now as $x^{-1} = (k\overline{k}^t)^{-1} = \overline{k}k^t = u\exp(-\varrho)\overline{u}^t$, we have

$$x^{-1}\,dx = \overline{k}\,(k^t\,dk + d\overline{k}^t\overline{k})\,\overline{k}^t = ue^{-\varrho/2}\,(e^{-\varrho/2}(\overline{u}^t\,du)e^{\varrho/2} + e^{-\varrho/2}(d\varrho)e^{\varrho/2} + e^{\varrho/2}(d\overline{u}^tu)e^{-\varrho/2})\,e^{\varrho/2}\overline{u}^t.$$

Noting that $d\overline{u}^tu = -\overline{u}^t\,du$, we take the volume form on both sides, denoted by the square brackets $[\cdot]$, to obtain

$$[k^t\,dk + \overline{(k^t\,dk)}^t] = [e^{-\varrho/2}(\overline{u}^t\,du)e^{\varrho/2} + e^{-\varrho/2}(d\varrho)e^{\varrho/2} - e^{\varrho/2}(\overline{u}^t\,du)e^{-\varrho/2}].$$

From the structure of u obtained in (4.13), it is easy to see that the invariant matrix differential form $\overline{u}^t\,du$ is of the form

$$\overline{u}^t\,du = \begin{pmatrix} \overline{v}^t\,dv & 0 \\ 0 & v^t\,d\overline{v} \end{pmatrix}.$$

Now, writing $e^{-\varrho/2}(\overline{u}^t\,du)e^{\varrho/2} + e^{-\varrho/2}(d\varrho)e^{\varrho/2} - e^{\varrho/2}(\overline{u}^t\,du)e^{-\varrho/2}$ in the familiar block decomposed form, we have

$$[k^t\,dk + \overline{(k^t\,dk)}^t] = \left[\begin{pmatrix} e^{-P/2} & 0 \\ 0 & e^{P/2} \end{pmatrix}\begin{pmatrix} \overline{v}^t\,dv & 0 \\ 0 & v^t\,d\overline{v} \end{pmatrix}\begin{pmatrix} e^{P/2} & 0 \\ 0 & e^{-P/2} \end{pmatrix} \right.$$
$$\left. - \begin{pmatrix} e^{P/2} & 0 \\ 0 & e^{-P/2} \end{pmatrix}\begin{pmatrix} \overline{v}^t\,dv & 0 \\ 0 & v^t\,d\overline{v} \end{pmatrix}\begin{pmatrix} e^{-P/2} & 0 \\ 0 & e^{P/2} \end{pmatrix} + \begin{pmatrix} dP & 0 \\ 0 & -dP \end{pmatrix} \right]$$

The right-hand side of the above equation gives

$$\left[\begin{pmatrix} dP + e^{-P/2}(\overline{v}^t\,dv)e^{P/2} - e^{P/2}(\overline{v}^t\,dv)e^{-P/2} & 0 \\ 0 & -dP + e^{P/2}(v^t\,d\overline{v})e^{-P/2} - e^{-P/2}(v^t\,d\overline{v})e^{P/2} \end{pmatrix} \right]$$

Now, taking $\overline{v}^t \, \mathrm{d}v = (\omega_{j,k})_{1 \le j,k \le n}$, we have

$$(\mathrm{d}P + e^{-P/2}(\overline{v}^t \, \mathrm{d}v)e^{P/2} - e^{P/2}(\overline{v}^t \, \mathrm{d}v)e^{-P/2})_{j,k} = \delta_{j,k} \, \mathrm{d}\varrho_j + 2\,\mathrm{sh}\big(\frac{\varrho_j - \varrho_k}{2}\big)\,\omega_{j,k}.$$

Therefore, we have

$$[k^t \, \mathrm{d}k + \overline{(k^t \, \mathrm{d}k)}^t] = c_n \prod_{1 \le j < k \le n} \mathrm{sh}^2\big(\frac{\varrho_j - \varrho_k}{2}\big) \bigwedge_{j=1}^{n} \mathrm{d}\varrho_j \bigwedge_{1 \le j < k \le n} (\omega_{j,k} \wedge \overline{\omega}_{j,k}).$$

Since $k^t \, \mathrm{d}k \in \mathfrak{k}$ is of the form

$$k^t \, \mathrm{d}k = \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \qquad (A, B \in \mathbb{C}^{n \times n}, B = B^t, A = -A^t),$$

we have

$$k^t \, \mathrm{d}k + \overline{(k^t \, \mathrm{d}k)}^t = 2i \begin{pmatrix} \mathrm{Im}(A) & \mathrm{Im}(B) \\ -\mathrm{Im}(B) & \mathrm{Im}(A) \end{pmatrix} = 2i\,\mathrm{Im}(k^t \, \mathrm{d}k).$$

Then, identifying $\mathrm{d}\mu(v) = [\overline{v}^t \, \mathrm{d}v] = \wedge_{1 \le j < k \le n}(\omega_{j,k} \wedge \overline{\omega}_{j,k})$, we have

$$\mathrm{d}\mu(k) = c_n \prod_{1 \le j < k \le n} \mathrm{sh}^2\big(\frac{\varrho_j - \varrho_k}{2}\big) \bigwedge_{j=1}^{n} \mathrm{d}\varrho_j \bigwedge \mathrm{d}\mu(v) \bigwedge \mathrm{d}\mu(k_0) \qquad (k_0 \in K_0).$$

This allows us to write

$$I_n(\kappa, t, 0_n) = c_n \frac{\exp\big(-\sum_{j=1}^{n} j^2 t/4\big)}{t^{n^2 + n/2}} \int_{\varrho_1 = 0}^{\infty} \cdots \int_{\varrho_n = 0}^{\infty} \frac{\varepsilon(\varrho) \exp\big(-\sum_{j=1}^{n}(\varrho_j^2/t - \kappa\varrho_j)\big)}{\delta(\varrho)} \times$$

$$\times \prod_{1 \le l < m \le n} \mathrm{sh}^2\big(\frac{\varrho_l - \varrho_m}{2}\big) \bigwedge_{j=1}^{n} \mathrm{d}\varrho_j.$$

Therefore, we have

$$\exp\big(-\frac{n\kappa}{4}(\kappa - (n+1))\,t\big)I_n(\kappa, t, 0_n) = c_n \int_{\varrho_1 = 0}^{\infty} \cdots \int_{\varrho_n = 0}^{\infty} \frac{\varepsilon(\varrho) \prod_{j=1}^{n} \exp\big(-(\varrho_j/\sqrt{t} - (\kappa - (n-j+1))\sqrt{t}/2)^2\big)}{t^{n^2 + n/2}} \times$$

$$\times \prod_{j=1}^{n} \frac{\exp(\varrho_j)}{\mathrm{sh}(\varrho_j)} \prod_{1 \le l < m \le n} \frac{\exp(\varrho_l)\,\mathrm{sh}\big((\varrho_l - \varrho_m)/2\big)}{\mathrm{sh}\big((\varrho_l + \varrho_m)/2\big)} \bigwedge_{j=1}^{n} \mathrm{d}\varrho_j$$

Now setting $\xi_j = \varrho_j/\sqrt{t} - (\kappa - (n-j+1))\sqrt{t}/2$, we have

$$\varrho_j = \xi_j \sqrt{t} + (\kappa - (n-j+1))t/2, \qquad (4.14)$$

whence one obtains

$$\bigwedge_{j=1}^{n} \mathrm{d}\varrho_j = t^{n/2} \bigwedge_{j=1}^{n} \mathrm{d}\xi_j.$$

Now see that

$$\lim_{t \to \infty} \frac{\varepsilon(\varrho)}{t^{n^2}} = \lim_{t \to \infty} \prod_{1 \le j \le n} \frac{\varrho_j}{t} \prod_{1 \le l < m \le n} \big(\frac{\varrho_l}{t} + \frac{\varrho_m}{t}\big) \prod_{1 \le l < m \le n} \big(\frac{\varrho_l}{t} - \frac{\varrho_m}{t}\big)$$

$$= \prod_{1 \le j \le n} \frac{\kappa - (n-j+1)}{2} \prod_{1 \le l < m \le n} \big(\kappa - \big(n - \frac{l+m}{2} + 1\big)\big) \prod_{1 \le l < m \le n} \frac{l - m}{2}.$$

Also, as $t \to \infty$, by the substitution (4.14), we have $\varrho_j \to \infty$ $(1 \le j \le n)$. Therefore, taking the limit at $t \to \infty$, we obtain

$$\lim_{t \to \infty} \frac{\exp(\varrho_j)}{\mathrm{sh}(\varrho_j)} = \lim_{\varrho_j \to \infty} \frac{\exp(\varrho_j)}{\mathrm{sh}(\varrho_j)} = 2.$$

Next, as

$$\varrho_l - \varrho_m = (\xi_l - \xi_m)\sqrt{t} + (l - m)t$$

and for $l < m$ we have $l - m < 0$, the quantity $\exp(\varrho_l - \varrho_m) \to 0$ as $t \to \infty$. Therefore, we have

$$\lim_{t \to \infty} \frac{\exp(\varrho_l)\,\mathrm{sh}\big((\varrho_l - \varrho_m)/2\big)}{\mathrm{sh}\big((\varrho_l + \varrho_m)/2\big)} = \lim_{t \to \infty} \frac{\exp(\varrho_l)\big(\exp\big((\varrho_l - \varrho_m)/2\big) - \exp\big(-(\varrho_l - \varrho_m)/2\big)\big)}{\exp\big((\varrho_l + \varrho_m)/2\big) - \exp\big(-(\varrho_l + \varrho_m)/2\big)}$$

$$= \lim_{t \to \infty} \frac{\exp(\varrho_l - \varrho_m) - 1}{1 - \exp(-(\varrho_l + \varrho_m))} = -1 \qquad (1 \le l < m \le n).$$

Combining all the above limits, we have

$$\lim_{t \to \infty} \exp\big(-\frac{n\kappa}{4}(\kappa - (n+1))\,t\big) I_n(\kappa, t, 0_n)$$

$$= c_n \prod_{1 \le j \le n} \frac{\kappa - (n - j + 1)}{2} \prod_{1 \le l < m \le n} \Big(\kappa - \big(n - \frac{l+m}{2} + 1\big)\Big) \prod_{1 \le l < m \le n} \frac{m - l}{2}$$

$$\le c_n \, \kappa^{n(n+1)/2}.$$

Then, from equation (4.8) and equation (4.10), the statement of the theorem easily follows. $\square$

Theorem 4.5. *For any arithmetic subgroup $\Gamma \subsetneq \mathrm{Sp}_n(\mathbb{R})$ such that $M := \Gamma \backslash \mathbb{H}_n$ is of finite volume, we have*

$$\sup_{Z \in \mathbb{H}_n} S_\kappa^\Gamma(Z) \le c_{n,\Gamma}\, \kappa^{3n(n+1)/4} \qquad (\kappa \ge n + 1),$$

where $c_{n,\Gamma}$ is a positive real constant depending only on n and Γ.

Proof. By Theorem 2.13, we know that the boundary $M^\star \setminus M$ of M consists of finite union of subspaces $M_j := (\Gamma \cap P(\mathbb{P}_j)) \backslash \mathbb{P}_j$, where $\mathbb{P}_j$ runs through a set of representatives of equivalence classes modulo Γ of rational boundary components of $\mathbb{H}_n$, and its subspaces of strictly smaller degree. We denote by $\mathcal{C}$ the set of all such inequivalent chains of boundary components of M. Then, for $\mathcal{P} \in \mathcal{C}$, we can define boundary neighbourhoods $U_\varepsilon(\mathcal{P})$ containing the entire chain $\mathcal{P}$, such that the complement of their union in M, i.e.,

$$K_\varepsilon := M \setminus \bigcup_{P \in \mathcal{C}} U_\varepsilon(\mathcal{P})$$

is a compact subset of M. We shall now estimate $S_\kappa^\Gamma(Z)$ for Z ranging through K_ε and $U_\varepsilon(\mathcal{P})\,(\mathcal{P} \in \mathcal{C})$, respectively.

In case of the compact set K_ε, using Theorem 4.3, we have already determined that

$$\sup_{Z \in K_\varepsilon} S_\kappa^\Gamma(Z) \le c_{n,\Gamma}\, \kappa^{n(n+1)/2} \qquad (\kappa \ge n + 1),$$

where the constant $c_{n,\Gamma} > 0$ depends only on n and Γ..

Next, in case of $U_\varepsilon(\mathcal{P})\,(\mathcal{P} \in \mathcal{C})$, by Remark 2.14, without loss of generality, we can assume $\mathcal{P}$ to be the chain

$$\Gamma_0 \backslash \mathbb{H}_0 < \Gamma_1 \backslash \mathbb{H}_1 < \ldots < \Gamma_j \backslash \mathbb{H}_j < \ldots < \Gamma_{n-1} \backslash \mathbb{H}_{n-1}$$

of standard boundary components of $\Gamma_n \backslash \mathbb{H}_n$.

Let $\mathscr{F}_n$ denote the standard fundamental domain of the Siegel modular group Γ_n. For $Z \in \mathscr{F}_n$, there exists a constant $c_3(n) > 0$ depending only on n, such that $Y \ge c_3(n)\mathbb{1}_n$ (see subsection 2.2). Let $\lambda_j(Y)\,(1 \le j \le n)$ denote the ordered set of eigenvalues

$$c_3(n) \le \lambda_1(Y) \le \ldots \le \lambda_j(Y) \le \ldots \le \lambda_n(Y)$$

of the positive definite matrix Y. Then $U_\varepsilon(\mathcal{P})$ can be taken as the neighbourhood

$$S_\varepsilon := \{Z = X + iY \in \mathscr{F}_n \mid \lambda_n(Y) > \varepsilon\}$$

of the standard boundary components of $\mathscr{F}_n$. As $\lambda_n(Y)$ denotes the highest eigenvalue of Y, the complement of S_ε in $\mathscr{F}_n$ is then given by the compact subset

$$K_\varepsilon = \{Z = X + iY \in \mathscr{F}_n \mid c_3(n)\mathbb{1}_n \leq Y \leq \varepsilon\mathbb{1}_n\}.$$

Let $f \in \mathcal{S}_\kappa^n(\Gamma)$ be a cusp form of weight κ. For $Z \in \mathbb{H}_n$ such that Y is Minkowski reduced and $Y > c\mathbb{1}_n$ for some $c > 0$, there exist positive numbers $c_1(n,c) > 0$ and $c_2(n,c) > 0$ depending only on n and c, such that

$$|f(Z)| \leq c_1(n,c)\,\exp(-c_2(n,c)\operatorname{tr}(Y)).$$

(see [26, page 57]). Since here we consider $Z \in \mathscr{F}_n$, we can take $c = c_3(n)$. In that case, the positive numbers $c_1(n,c) > 0$ and $c_2(n,c) > 0$ depend only on n and we have

$$|f(Z)| \leq c_1(n)\,\exp(-c_2(n)\operatorname{tr}(Y)) \qquad (Z \in \mathscr{F}_n). \tag{4.15}$$

This shows that the function $f(Z)/\exp(ic_2(n)\operatorname{tr}(Z))$ is a bounded holomorphic function on S_ε and hence, by maximum modulus principle, its absolute value

$$\left|\frac{f(Z)}{\exp\left(ic_2(n)\operatorname{tr}(Z)\right)}\right|^2 = exp(2c_2(n)\operatorname{tr}(Y))|f(Z)|^2$$

takes its maximum value at the boundary

$$\partial S_\varepsilon = \{Z = X + iY \in \mathscr{F}_n \mid \lambda_n(Y) = \varepsilon\}$$

of S_ε. Now, write $\det(Y)^\kappa|f(Z)|^2$ as

$$\det(Y)^\kappa|f(Z)|^2 = \exp(2c_2(n)\operatorname{tr}(Y))|f(Z)|^2\,\frac{\det(Y)^\kappa}{\exp(2c_2(n)\operatorname{tr}(Y))}.$$

Then writing the eigenvalues of Y as $\lambda_j(Y)\,(1 \leq j \leq n)$, we have

$$\frac{\det(Y)^\kappa}{\exp(2c_2(n)\operatorname{tr}(Y))} = \prod_{j=1}^n \frac{\lambda_j(Y)^\kappa}{\exp(2c_2(n)\lambda_j(Y))}.$$

The functions $\lambda_j^\kappa/\exp(2c_2(n)\lambda_j)$ attain maxima at $\lambda_j = \kappa/(2c_2(n))$ and decreases monotonically for $\lambda_j > \kappa/(2c_2(n))$. Therefore, if we choose $\varepsilon > \kappa/(2c_2(n))$, then we have

$$\sup_{Z \in M} S_\kappa^\Gamma(Z) = \sup_{Z \in K_\varepsilon} S_\kappa^\Gamma(Z) \leq c_{n,\Gamma}\,\kappa^{n(n+1)/2} \qquad \left(\kappa \geq n + 1,\, \varepsilon > \frac{\kappa}{2c_2(n)}\right).$$

Now, in case $\varepsilon \leq \kappa/(2c_2(n))$, we need to determine $\sup_{Z \in M} S_\kappa^\Gamma(Z)$ in the annulus

$$S_\varepsilon \setminus S_{\kappa/(2c_2(n))} = \left\{Z = X + iY \in \mathscr{F}_n \;\middle|\; \varepsilon < \lambda_n(Y) \leq \frac{\kappa}{2c_2(n)}\right\}$$

$$\subsetneq \left\{Z = X + iY \in \mathscr{F}_n \;\middle|\; Y \leq \frac{\kappa}{2c_2(n)}\mathbb{1}_n\right\}.$$

We do this using Theorems 4.3 and 4.4.

From equation (4.8), we have

$$S_\kappa^\Gamma(Z) = \lim_{t\to\infty} \exp\left(-\frac{n\kappa}{4}(\kappa - (n+1))\,t\right)\sum_{\gamma\in\Gamma} K_t^{(\kappa)}(2R^\gamma(Z)).$$

We split the sum over Γ according as whether there is a minimum distance between the point γZ and Z or they can get arbitrarily close. Let Γ_∞ denote the set of elements of Γ for which γZ and Z can get arbitrarily close. Then we split the above sum as

$$S_\kappa^\Gamma(Z) = \lim_{t\to\infty} \exp\left(-\frac{n\kappa}{4}(\kappa - (n+1))\,t\right)\sum_{\gamma\in\Gamma\setminus\Gamma_\infty} K_t^{(\kappa)}(2R^\gamma(Z))$$

$$+ \lim_{t\to\infty} \exp\left(-\frac{n\kappa}{4}(\kappa - (n+1))\,t\right)\sum_{\gamma\in\Gamma_\infty} K_t^{(\kappa)}(2R^\gamma(Z)).$$

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As the function $\exp\left(-\frac{n\kappa}{4}(\kappa-(n+1))t\right)\sum_{\gamma\in\Gamma\backslash\Gamma_\infty}K_t^{(\kappa)}(2R^\gamma(Z))$ is monotonically decreasing in t, we have

$$
\begin{aligned}
S_\kappa^\Gamma(Z) \leq {} & \exp\left(-\frac{n\kappa}{4}(\kappa-(n+1))t\right)\sum_{\gamma\in\Gamma\backslash\Gamma_\infty}K_t^{(\kappa)}(2R^\gamma(Z)). \\
& + \lim_{t\to\infty}\exp\left(-\frac{n\kappa}{4}(\kappa-(n+1))t\right)\sum_{\gamma\in\Gamma_\infty}K_t^{(\kappa)}(2R^\gamma(Z)).
\end{aligned}
\tag{4.16}
$$

As for $\gamma\in\Gamma\backslash\Gamma_\infty$ the points γZ and Z cannot be arbitrarily close, the first sum can be handled exactly as in Theorem 4.3 using the counting function to estimate the sum by an integral to give

$$
\exp\left(-\frac{n\kappa}{4}(\kappa-(n+1))t\right)\sum_{\gamma\in\Gamma\backslash\Gamma_\infty}K_t^{(\kappa)}(2R^\gamma(Z)) \leq c_{n,\Gamma}\,\kappa^{n(n+1)/2}.
\tag{4.17}
$$

The second sum was estimated in Theorem 4.4 to be

$$
\lim_{t\to\infty}\exp\left(-\frac{n\kappa}{4}(\kappa-(n+1))t\right)\sum_{\gamma\in\Gamma_\infty}K_t^{(\kappa)}(2R^\gamma(Z)) \leq c_n\,\kappa^{n(n+1)/2}\sum_{\gamma\in\Gamma_\infty}\frac{1}{\prod_{j=1}^n\,\mathrm{ch}^\kappa(r_j^\gamma(Z))}.
\tag{4.18}
$$

Thus, it only remains to estimate the sum

$$
\sum_{\gamma\in\Gamma_\infty}\frac{1}{\prod_{j=1}^n\,\mathrm{ch}^\kappa(r_j^\gamma(Z))}.
\tag{4.19}
$$

Since Γ_∞ is defined as the set of elements of Γ for which γZ and Z can get arbitrarily close, by Remark 2.12, we have

$$
\Gamma_\infty = \bigcup_{j=0}^{n-1}\Gamma_\infty^j,
$$

where $\Gamma_\infty^j := \Gamma\cap W_j$. Thus, by (2.9), these groups are explicitly given by

$$
\begin{aligned}
\Gamma_\infty^0 &= \left\{\begin{pmatrix}\mathbb{1}_n & S \\ 0 & \mathbb{1}_n\end{pmatrix}\,\middle|\, S\in\mathrm{Sym}_n(\mathbb{Z})\right\}, \\
\Gamma_\infty^j &= \left\{\begin{pmatrix}A & AS \\ 0 & A^{-t}\end{pmatrix}\,\middle|\, A=\begin{pmatrix}\mathbb{1}_j & 0 \\ L & \mathbb{1}_{n-j}\end{pmatrix}, S=\begin{pmatrix}0 & H^t \\ H & S_2\end{pmatrix}\right\} \qquad (1\leq j\leq(n-1)),
\end{aligned}
$$

where $L, H\in\mathbb{Z}^{(n-j)\times j}$ and $S_2\in\mathrm{Sym}_{n-j}(\mathbb{Z})$.

Next, we need an effective way of calculating the quantity $1/\prod_{j=1}^n\,\mathrm{ch}^\kappa(r_j^\gamma(Z))$. Here we derive a more general formula for the quantity $1/\prod_{j=1}^n\,\mathrm{ch}^2(r_j(Z,W))$ $(Z,W\in\mathbb{H}_n)$, where setting $W=\gamma Z\,(\gamma\in\Gamma_\infty)$, we can easily obtain the sum in (4.19) above.

Recall from subsection 2.1 the cross ratio

$$
\rho(W,Z) = (W-Z)(\overline{W}-Z)^{-1}(\overline{W}-\overline{Z})(W-\overline{Z})^{-1} \quad (Z,W\in\mathbb{H}_n).
$$

Let $\rho_j(Z,W)\,(1\leq j\leq n)$ denote the eigenvalues of $\rho(W,Z)$. The point $Z=X+iY\in\mathbb{H}_n$, where $X,Y\in\mathbb{R}^{n\times n}$ with $Y>0$ can be written as

$$
Z = \begin{pmatrix}\mathbb{1}_n & X \\ 0 & \mathbb{1}_n\end{pmatrix}\begin{pmatrix}Y^{1/2} & 0 \\ 0 & Y^{-1/2}\end{pmatrix}\cdot i\mathbb{1}_n.
$$

Now, as the matrices $\rho(Z,W)$ and $\rho(gZ,gW)$ have the same set of eigenvalues for all $g\in\mathrm{Sp}_n(\mathbb{R})$, setting

$$
V = \begin{pmatrix}Y^{-1/2} & 0 \\ 0 & Y^{1/2}\end{pmatrix}\begin{pmatrix}\mathbb{1}_n & -X \\ 0 & \mathbb{1}_n\end{pmatrix}\cdot W = Y^{-1/2}(W-X)Y^{-1/2},
\tag{4.20}
$$

the cross ratio $\rho(V,i\mathbb{1}_n)$ has the same eigenvalues as $\rho(Z,W)$, i.e., $\rho_j(Z,W)\,(1\leq j\leq n)$. Therefore, we have

$$
\det(\mathbb{1}_n-\rho(Z,W)) = \det\left(\mathbb{1}_n-(V-i\mathbb{1}_n)(V+i\mathbb{1}_n)^{-1}(\overline{V}+i\mathbb{1}_n)(\overline{V}-i\mathbb{1}_n)^{-1}\right).
$$

Since these eigenvalues are of the form

$$\rho_j(Z, W) = \operatorname{th}^2(r_j(Z, W)) \qquad (1 \le j \le n),$$

from the above equations, using the fact $(V - i\mathbb{1}_n)(V + i\mathbb{1}_n)^{-1} = (V + i\mathbb{1}_n)^{-1}(V - i\mathbb{1}_n)$, one obtains

$$\frac{1}{\prod_{j=1}^n \operatorname{ch}^2(r_j(Z, W))} = \frac{\det\left((V + i\mathbb{1}_n)(\overline{V} - i\mathbb{1}_n) - (V - i\mathbb{1}_n)(\overline{V} + i\mathbb{1}_n)\right)}{\det(V + i\mathbb{1}_n)\det(\overline{V} - i\mathbb{1}_n)}$$

$$= \frac{\det(2i(\overline{V} - V))}{\det(V + i\mathbb{1}_n)\det(\overline{V} - i\mathbb{1}_n)}.$$

Then, using the definition of V in equation (4.20), one obtains

$$\frac{1}{\prod_{j=1}^n \operatorname{ch}^2(r_j(Z, W))} = \frac{4^n \det(\operatorname{Im}(Z)) \det(\operatorname{Im}(W))}{|\det(W - \overline{Z})|^2}. \tag{4.21}$$

Next we estimate the sum in (4.19) by breaking the sum over Γ_∞ into sums over Γ_∞^j ($0 \le j \le (n-1)$). We begin with Γ_∞^0. For $\gamma \in \Gamma_\infty^0$, i.e.,

$$\gamma = \begin{pmatrix} \mathbb{1}_n & S \\ 0 & \mathbb{1}_n \end{pmatrix} \qquad (S \in \operatorname{Sym}_n(\mathbb{Z}))$$

we have $\gamma Z = Z + S$. Therefore, putting $W = Z + S$ in equation (4.21), we obtain

$$\frac{1}{\prod_{j=1}^n \operatorname{ch}^2(r_j^\gamma(Z))} = \frac{4^n \det(Y)^2}{\det(S - 2iY)\det(S + 2iY)} = \frac{1}{\det(\mathbb{1}_n + (\frac{1}{2}Y^{-1/2}SY^{-1/2})^2)}.$$

Then we estimate the sum over Γ_∞^0 by the matrix beta integral

$$\sum_{\gamma \in \Gamma_\infty^0} \frac{1}{\prod_{j=1}^n \operatorname{ch}^\kappa(r_j^\gamma(Z))} \le \int_{S \in \operatorname{Sym}_n(\mathbb{R})} \frac{[dS]}{\det(\mathbb{1}_n + (\frac{1}{2}Y^{-1/2}SY^{-1/2})^2)^{\kappa/2}} \tag{4.22}$$

Now, setting $T = \frac{1}{2}Y^{-1/2}SY^{-1/2}$, we have

$$[dT] = c_n \det(Y)^{-(n+1)/2}[dS].$$

This gives us

$$\sum_{\gamma \in \Gamma_\infty^0} \frac{1}{\prod_{j=1}^n \operatorname{ch}^\kappa(r_j^\gamma(Z))} \le c_n \det(Y)^{(n+1)/2} \int_{T \in \operatorname{Sym}_n(\mathbb{R})} \frac{[dT]}{\det(\mathbb{1}_n + T^2)^{\kappa/2}}.$$

Then, using Hua's matrix beta integral (see [22, page 33])

$$\int_{T \in \operatorname{Sym}_n(\mathbb{R})} \frac{[dT]}{\left(\det\left(I + T^2\right)\right)^\alpha} = \pi^{n(n+1)/4} \frac{\Gamma\left(\alpha - n/2\right)}{\Gamma(\alpha)} \prod_{\nu=1}^{n-1} \frac{\Gamma\left(2\alpha - (n + \nu)/2\right)}{\Gamma(2\alpha - \nu)} \qquad (\alpha > n/2), \tag{4.23}$$

and $\det(Y) < (\kappa/(2c_2(n)))^n$, from the above calculations, it easily follows that

$$\sum_{\gamma \in \Gamma_\infty^0} \frac{1}{\prod_{j=1}^n \operatorname{ch}^\kappa(r_j^\gamma(Z))} \le c_n \kappa^{n(n+1)/4}. \tag{4.24}$$

Next we consider the sum

$$\sum_{\gamma \in \Gamma_\infty^j} \frac{1}{\prod_{j=1}^n \operatorname{ch}^\kappa(r_j^\gamma(Z))} \qquad (1 \le j \le (n-1)).$$

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For $\gamma \in \Gamma_\infty^j$ $(1 \le j \le (n-1))$, i.e.,

$$\gamma = \begin{pmatrix} A & AS \\ 0 & A^{-t} \end{pmatrix} \qquad \left(A = \begin{pmatrix} \mathbb{1}_j & 0 \\ L & \mathbb{1}_{n-j} \end{pmatrix}, \ S = \begin{pmatrix} 0 & H^t \\ H & S_2 \end{pmatrix} \right),$$

where $L, H \in \mathbb{Z}^{(n-j) \times j}$ and $S_2 \in \mathbb{Z}^{(n-j) \times (n-j)}$, $S_2 = S_2^t$, we have $\gamma Z = A(Z+S)A^t$. Therefore, putting $W = A(Z+S)A^t$ in equation (4.21), we obtain

$$\frac{1}{\prod_{j=1}^n \mathrm{ch}^2(r_j^\gamma(Z))} = \frac{4^n \det(Y)^2}{|\det(A(Z+S)A^t - \overline{Z})|^2}$$

$$= \frac{4^n \det(Y)^2}{\left| \det\left((A(X+S)A^t - X) + i(AYA^t + Y) \right) \right|^2}$$

Now, just as in the $j = 0$ case above, we estimate the sum over Γ_∞^j by a matrix integral $I_{n,\kappa}(Z)$, i.e.,

$$\sum_{\gamma \in \Gamma_\infty^j} \frac{1}{\prod_{j=1}^n \mathrm{ch}^\kappa(r_j^\gamma(Z))} \le I_{n,\kappa}^j(Z),$$

where the integral $I_{n,\kappa}(Z)$ is given by

$$I_{n,\kappa}^j(Z) = \int\limits_L \int\limits_H \int\limits_{S_2} \frac{2^{n\kappa} \det(Y)^\kappa \ [dS_2] \wedge [dH] \wedge [dL]}{\left| \det\left((A(X+S)A^t - X) + i(AYA^t + Y) \right) \right|^\kappa}.$$

Next consider the the block decomposition

$$X = \begin{pmatrix} X_1 & X_{12}^t \\ X_{12} & X_2 \end{pmatrix} \qquad (X_1 \in \mathbb{R}^{j \times j}, \ X_2 \in \mathbb{R}^{(n-j) \times (n-j)}, \ X_{12} \in \mathbb{R}^{(n-j) \times j}).$$

of the matrix $X \in \mathbb{R}^{n \times n}$. Then, we have

$$A(X+S)A^t - X = \begin{pmatrix} 0 & H^t + X_1 L^t \\ H + LX_1 & S_2 + (LH^t + HL^t) + (LX_{12}^t + X_{12}L^t + LX_1L^t) \end{pmatrix}.$$

Now, since

$$[d(S_2 + (LH^t + HL^t) + (LX_{12}^t + X_{12}L^t + LX_1L^t))] \wedge [d(H + LX_1)] \wedge [dL]$$
$$= [dS_2] \wedge [dH] \wedge [dL],$$

we can simply replace the term $A(X+S)A^t - X$ in $I_{n,\kappa}(Z)$ with S, to write

$$I_{n,\kappa}^j(Z) = \int\limits_L \int\limits_H \int\limits_{S_2} \frac{2^{n\kappa} \det(Y)^\kappa \ [dS_2] \wedge [dH] \wedge [dL]}{|\det((AYA^t + Y) + iS)|^\kappa}.$$

Next we write the positive definite matrix $Y > 0$ in the Cholesky decomposed form $Y = BB^t$, where

$$B = \begin{pmatrix} P_1 & 0 \\ P & P_2 \end{pmatrix} \qquad (P_1 \in \mathbb{R}^{j \times j}, \ P_2 \in \mathbb{R}^{(n-j) \times (n-j)}, \ P \in \mathbb{R}^{(n-j) \times j})$$

with P_1, P_2 non-singular lower triangular. Then we have

$$I_{n,\kappa}^j(Z) = \int\limits_L \int\limits_H \int\limits_{S_2} \frac{[dS_2] \wedge [dH] \wedge [dL]}{\left| \det\left(1/2(\mathbb{1}_n + (B^{-1}AB)(B^{-1}AB)^t + iB^{-1}SB^{-t}) \right) \right|^\kappa}. \tag{4.25}$$

The matrices $B^{-1}AB$ and $B^{-1}SB^{-t}$, in block decomposed form, are given by

$$B^{-1}AB = \begin{pmatrix} \mathbb{1}_j & 0 \\ P_2^{-1}LP_1 & \mathbb{1}_{n-j} \end{pmatrix},$$

$$B^{-1}SB^{-t} = \begin{pmatrix} 0 & P_1^{-1}H^tP_2^{-t} \\ P_2^{-1}HP_1^{-t} & P_2^{-1}S_2P_2^{-t} - P_2^{-1}(HP_1^{-t}P^t + PP_1^{-1}H^t)P_2^{-t} \end{pmatrix},$$

respectively. We set

$$2\widetilde{L} = P_2^{-1}LP_1, \tag{4.26}$$

$$2\widetilde{H} = P_2^{-1}HP_1^{-t}, \tag{4.27}$$

$$2\widetilde{S_2} = P_2^{-1}S_2P_2^{-t} - P_2^{-1}(HP_1^{-t}P^t + PP_1^{-1}H^t)P_2^{-t} \tag{4.28}$$

Then the matrix $1/2(\mathbb{1}_n + (B^{-1}AB)(B^{-1}AB)^t + iB^{-1}SB^{-t})$ in the denominator of the integrand in equation (4.25) is given by

$$\frac{1}{2}(\mathbb{1}_n + (B^{-1}AB)(B^{-1}AB)^t + iB^{-1}SB^{-t}) = \begin{pmatrix} \mathbb{1}_j & \widetilde{L}^t + i\widetilde{H}^t \\ \widetilde{L} + i\widetilde{H} & \mathbb{1}_{n-j} + 2\widetilde{L}\widetilde{L}^t + i\widetilde{S_2} \end{pmatrix}$$

and the corresponding determinant is given by

$$\det\left(\frac{1}{2}(\mathbb{1}_n + (B^{-1}AB)(B^{-1}AB)^t + iB^{-1}SB^{-t})\right)$$
$$= \det((\mathbb{1}_{n-j} + \widetilde{L}\widetilde{L}^t + \widetilde{H}\widetilde{H}^t) + i(\widetilde{S_2} - \widetilde{H}\widetilde{L}^t - \widetilde{L}\widetilde{H}^t)).$$

Next we set

$$Q = \mathbb{1}_{n-j} + \widetilde{L}\widetilde{L}^t + \widetilde{H}\widetilde{H}^t,$$
$$T = \widetilde{S_2} - \widetilde{H}\widetilde{L}^t - \widetilde{L}\widetilde{H}^t.$$

Then the integral $I_{n,\kappa}^j(Z)$ in equation (4.25) is given by

$$I_{n,\kappa}^j(Z) = \int_{S_2}\int_H\int_L \frac{[\mathrm{d}S_2] \wedge [\mathrm{d}H] \wedge [\mathrm{d}L]}{|\det(Q + iT)|^\kappa}.$$

Next we need to calculate the volume form $[\mathrm{d}S_2] \wedge [\mathrm{d}H] \wedge [\mathrm{d}L]$ in terms of $T, \widetilde{L}$ and $\widetilde{H}$. From equations (4.26) and (4.27), we obtain

$$2^{j(n-j)}[\mathrm{d}\widetilde{L}] = \frac{\det(P_1)^{n-j}}{\det(P_2)^j}[\mathrm{d}L],$$
$$2^{j(n-j)}[\mathrm{d}\widetilde{H}] = \frac{1}{\det(P_1)^{n-j}\det(P_2)^j}[\mathrm{d}H].$$

From equation (4.28), one obtains

$$2^{(n-j)(n-j+1)/2}[\mathrm{d}\widetilde{S_2}] \wedge [\mathrm{d}H] \wedge [\mathrm{d}L] = \frac{[\mathrm{d}S_2] \wedge [\mathrm{d}H] \wedge [\mathrm{d}L]}{\det(P_2)^{n-j+1}}.$$

Now, since $[\mathrm{d}\widetilde{S_2}] \wedge [\mathrm{d}\widetilde{H}] \wedge [\mathrm{d}\widetilde{L}] = [\mathrm{d}T] \wedge [\mathrm{d}\widetilde{H}] \wedge [\mathrm{d}\widetilde{L}]$, we conclude that

$$[\mathrm{d}S_2] \wedge [\mathrm{d}H] \wedge [\mathrm{d}L] = 2^{(n-j)(n-j+1)/2+2j(n-j)}\det(P_2)^{n-j+1}\det(P_2)^{2j}[\mathrm{d}T] \wedge [\mathrm{d}\widetilde{H}] \wedge [\mathrm{d}\widetilde{L}].$$

Therefore, we have

$$I_{n,\kappa}^j(Z) \le c_n \det(P_2)^{n-j+1}\det(P_2)^{2j}\int_{\widetilde{L}}\int_{\widetilde{H}}\int_T \frac{[\mathrm{d}T] \wedge [\mathrm{d}\widetilde{H}] \wedge [\mathrm{d}\widetilde{L}]}{|\det(Q + iT)|^\kappa}, \tag{4.29}$$

where c_n, as usual, stands for a generic constant depending only on n. As the matrix $Q = \mathbb{1}_{n-j} + \widetilde{L}\widetilde{L}^t + \widetilde{H}\widetilde{H}^t$ is positive definite, the integral

$$\int_{T\in\mathrm{Sym}_{n-j}(\mathbb{R})} \frac{[\mathrm{d}T]}{|\det(Q + iT)|^\kappa}$$

can be written as

$$\int\limits_{T\in\mathrm{Sym}_{n-j}(\mathbb{R})} \frac{[\mathrm{d}T]}{|\det(Q+iT)|^\kappa} = \frac{1}{\det(Q)^\kappa} \int\limits_{T\in\mathrm{Sym}_{n-j}(\mathbb{R})} \frac{[\mathrm{d}T]}{|\det(\mathbb{1}_{n-j}+iQ^{-1/2}TQ^{-1/2})|^\kappa}.$$

Then setting $\widetilde{T} = Q^{-1/2}TQ^{-1/2}$, we have

$$[\mathrm{d}T] = \det(Q)^{(n-j+1)/2}[\mathrm{d}\widetilde{T}].$$

Then, using the Hua integral in (4.23), we obtain

$$\int\limits_{T\in\mathrm{Sym}_{n-j}(\mathbb{R})} \frac{[\mathrm{d}T]}{|\det(Q+iT)|^\kappa} = \frac{1}{\det(Q)^{\kappa-(n-j+1)/2}} \int\limits_{\widetilde{T}=\widetilde{T}^t} \frac{[\mathrm{d}\widetilde{T}]}{\det(\mathbb{1}_n+\widetilde{T}^2)^{\kappa/2}}$$

$$\leq c_n \frac{\kappa^{-(n-j)(n-j+1)/4}}{\det(Q)^{\kappa-(n-j+1)/2}}. \tag{4.30}$$

Also, from $Y \leq (\kappa/2c_2(n))\mathbb{1}_n$, i.e.,

$$Y = BB^t = \begin{pmatrix} P_1 & 0 \\ P & P_2 \end{pmatrix}\begin{pmatrix} P_1^t & P^t \\ 0 & P_2^t \end{pmatrix} = \begin{pmatrix} P_1P_1^t & P_1P^t \\ PP_1^t & P_2P_2^t + PP^t \end{pmatrix} \leq \frac{\kappa}{2c_2(n)}\mathbb{1}_n,$$

one obtains that

$$P_2P_2^t + PP^t - PP_1^t(P_1P_1^t)^{-1}P_1P^t = P_2P_2^t \leq \frac{\kappa}{2c_2(n)}\mathbb{1}_{n-j}. \tag{4.31}$$

Thus we have $\det(P_2) \leq c_n\kappa^{(n-j)/2}$. Hence, by equation (4.29) and equation (4.30), we have the estimate

$$I_{n,\kappa}^j(Z) \leq c_n \, \kappa^{-(n-j)(n-j+1)/4} \, \kappa^{(n-j)(n-j+1)/2} \, \kappa^{j(n-j)}$$

$$\cdot \int\limits_{\widetilde{L}} \int\limits_{\widetilde{H}} \frac{[\mathrm{d}\widetilde{H}] \wedge [\mathrm{d}\widetilde{L}]}{\det(\mathbb{1}_{n-j}+\widetilde{L}\widetilde{L}^t+\widetilde{H}\widetilde{H}^t)^{\kappa-(n-j+1)/2}}.$$

Now, to estimate the the integral

$$\int\limits_{\widetilde{L}} \int\limits_{\widetilde{H}} \frac{[\mathrm{d}\widetilde{H}] \wedge [\mathrm{d}\widetilde{L}]}{\det(\mathbb{1}_{n-j}+\widetilde{L}\widetilde{L}^t+\widetilde{H}\widetilde{H}^t)^{\kappa-(n-j+1)/2}},$$

we set

$$\mathbb{1}_{n-j} + \widetilde{L}\widetilde{L}^t = EE^t, \quad E^{-1}\widetilde{H} = U.$$

Then, the above integral splits as

$$\int\limits_{\widetilde{L}} \int\limits_{\widetilde{H}} \frac{[\mathrm{d}\widetilde{H}] \wedge [\mathrm{d}\widetilde{L}]}{\det(\mathbb{1}_{n-j}+\widetilde{L}\widetilde{L}^t+\widetilde{H}\widetilde{H}^t)^{\kappa-(n-j+1)/2}}$$

$$= \int\limits_{\widetilde{L}\in\mathbb{R}^{(n-j)\times j}} \frac{[\mathrm{d}\widetilde{L}]}{\det(\mathbb{1}_{n-j}+\widetilde{L}\widetilde{L}^t)^{\kappa-(n-j)-1/2}} \int\limits_{U\in\mathbb{R}^{(n-j)\times j}} \frac{[\mathrm{d}U]}{\det(\mathbb{1}_{n-j}+UU^t)^{\kappa-(n-j+1)/2}}.$$

Proceeding as in [22, Theorem 2.2.1], for matrices $X \in \mathbb{R}^{p\times q}$ ($p, q \in \mathbb{N}_{\geq 1}$) one obtains the formula

$$\int\limits_{X\in\mathbb{R}^{p\times q}} \frac{[\mathrm{d}X]}{\det(\mathbb{1}_p+XX^t)^\mu} = \pi^{pq/2}\prod_{l=1}^q \frac{\Gamma(\mu-(l-1)/2-p/2)}{\Gamma(\mu-(l-1)/2)} \qquad (\mu > (p+q-1)/2).$$

Using this formula, it immediately follows that

$$\int\limits_{U\in\mathbb{R}^{(n-j)\times j}} \frac{[\mathrm{d}U]}{\det(\mathbb{1}_{n-j}+UU^t)^{\kappa-(n-j+1)/2}} \leq c_n\kappa^{-j(n-j)/2},$$

$$\int\limits_{\widetilde{L}\in\mathbb{R}^{(n-j)\times j}} \frac{[\mathrm{d}\widetilde{L}]}{\det(\mathbb{1}_{n-j}+\widetilde{L}\widetilde{L}^t)^{\kappa-(n-j)-1/2}} \leq c_n\kappa^{-j(n-j)/2}.$$

Thus, we have the estimate for the integral

$$\underset{\widetilde{L}}{\int} \underset{\widetilde{H}}{\int} \frac{[\mathrm{d}\widetilde{H}] \wedge [\mathrm{d}\widetilde{L}]}{\det(\mathbb{1}_{n-j} + \widetilde{L}\widetilde{L}^t + \widetilde{H}\widetilde{H}^t)^{\kappa - (n-j+1)/2}} \leq c_n \kappa^{-j(n-j)},$$

thereby giving

$$I_{n,\kappa}^{j}(Z) \leq c_n \, \kappa^{(n-j)(n-j+1)/4}.$$

Hence, we have that

$$\sum_{\gamma \in \Gamma_\infty^j} \frac{1}{\prod_{j=1}^n \mathrm{ch}^\kappa(r_j^\gamma(Z))} \leq c_n \, \kappa^{(n-j)(n-j+1)/4} \qquad (0 \leq j \leq (n-1))$$

and consequently,

$$\sum_{\gamma \in \Gamma_\infty} \frac{1}{\prod_{j=1}^n \mathrm{ch}^\kappa(r_j^\gamma(Z))} \leq c_n \, \kappa^{n(n+1)/4}.$$

Thus, from equation (4.18), we have

$$\lim_{t \to \infty} \exp\left(-\frac{n\kappa}{4}(\kappa - (n+1))\,t\right) \sum_{\gamma \in \Gamma_\infty} K_t^{(\kappa)}(2R^\gamma(Z)) \leq c_n \, \kappa^{3n(n+1)/4},$$

whence the theorem follows. □

4.3   Uniform sup-norm bounds

Theorem 4.6. *Let $\Gamma_0 \subsetneq \mathrm{Sp}_n(\mathbb{R})$ be a fixed arithmetic subgroup of $\mathrm{Sp}_n(\mathbb{R})$ such that $M_0 := \Gamma_0 \backslash \mathbb{H}_n$ is of finite volume. Let $\Gamma \subseteq \Gamma_0$ a subgroup of finite index. Then, for $\kappa \geq n + 1$, we have*

$$\sup_{Z \in \mathbb{H}_n} S_\kappa^\Gamma(Z) \leq c_{n,\Gamma_0} \, \kappa^{3n(n+1)/4} \qquad (\kappa \geq n+1),$$

where c_{n,Γ_0} is a positive real constant depending only on n and Γ_0.

Proof. As in the proof of Theorem 4.5, we denote by $\mathcal{C}_0$ the set of all inequivalent chains of boundary components of M_0 and choose boundary neighbourhoods $U_\varepsilon(\mathcal{P}_0)\,(\mathcal{P}_0 \in \mathcal{C}_0)$ containing the entire chain $\mathcal{P}_0$ such that the complement of their union in M_0, i.e.,

$$K_{0,\varepsilon} := M_0 \setminus \bigcup_{P_0 \in \mathcal{C}_0} U_\varepsilon(\mathcal{P}_0)$$

is a compact subset of M_0.

Let $M := \Gamma \backslash \mathbb{H}_n$ and $\pi \colon M \to M_0$ denote the covering map. Then by means of $K_{0,\varepsilon}$, we obtain the compact subset $K_\varepsilon := \pi^{-1}(K_{0,\varepsilon})$ of M. Since $\Gamma \subseteq \Gamma_0$, from equation (4.4), by expanding the sum over Γ to that over the larger group Γ_0, we have

$$S_\kappa^\Gamma(Z) \leq \exp\left(-\frac{n\kappa}{4}(\kappa - (n+1))\,t\right) \sum_{\gamma \in \Gamma} K_t^{(\kappa)}(2R^\gamma(Z))$$

$$\leq \exp\left(-\frac{n\kappa}{4}(\kappa - (n+1))\,t\right) \sum_{\gamma \in \Gamma_0} K_t^{(\kappa)}(2R^\gamma(Z)),$$

which, by Theorem 4.3, gives the uniform bound

$$\sup_{Z \in K_\varepsilon} S_\kappa^\Gamma(Z) \leq c_{n,\Gamma_0} \, \kappa^{n(n+1)/2} \qquad (\kappa \geq n+1). \tag{4.32}$$

We are thus left to bound the quantity $S_\kappa^\Gamma(Z)$ in the neighbourhoods of M obtained by pulling back the neighbourhoods $U_\varepsilon(\mathcal{P}_0)$ $(\mathcal{P}_0 \in \mathcal{C}_0)$ of M_0 to M. In order to do this, as in the proof of Theorem 4.5, we can again assume without loss of generality that $\mathcal{P}_0$ is the chain

$$\Gamma_0\backslash\mathbb{H}_0 < \Gamma_1\backslash\mathbb{H}_1 < \ldots < \Gamma_j\backslash\mathbb{H}_j < \ldots < \Gamma_{n-1}\backslash\mathbb{H}_{n-1}$$

of standard boundary components of $\Gamma_n\backslash\mathbb{H}_n$. Furthermore, we may also assume that the chain $\mathcal{P} \in \mathcal{C}$ of boundary components of M lying over $\mathcal{P}_0$ is also the chain of standard boundary components of $\Gamma_n\backslash\mathbb{H}_n$ of ramification index ℓ, say. Then a cusp form $f \in \mathcal{S}_\kappa^n(\Gamma)$ of weight κ has a Fourier expansion (see equation (2.12))

$$f(Z) = \sum_{\substack{T \in \mathrm{Sym}_n(\mathbb{Q}),\, T > 0 \\ T \text{ half-integral}}} a(T) \exp\left(\frac{2\pi i}{\ell}\, \mathrm{tr}(TZ)\right)$$

at $\mathcal{P}$. Then, just like in equation (4.15) in Theorem 4.5, we obtain positive numbers $c_1(n) > 0$ and $c_2(n) > 0$ depending only on n such that

$$|f(Z)| \leq c_1(n)\, \exp(-c_2(n)\, \mathrm{tr}(Y)/\ell) \qquad (Z \in \mathscr{F}_n).$$

Then proceeding as in Theorem 4.5 with $c_2(n)$ replaced by $c_2(n)/\ell$, one sees that for $\varepsilon > \kappa\ell/(2c_2(n))$, we have

$$\sup_{Z \in M} S_\kappa^\Gamma(Z) = \sup_{Z \in K_\varepsilon} S_\kappa^\Gamma(Z) \qquad \left(\varepsilon > \frac{\kappa\ell}{2c_2(n)}\right),$$

which, by equation (4.32) gives the uniform estimate

$$\sup_{Z \in M} S_\kappa^\Gamma(Z) \leq c_{n,\Gamma_0}\, \kappa^{n(n+1)/2} \qquad (\kappa \geq n+1).$$

Thus, we are left only to bound $S_\kappa^\Gamma(Z)$ in the range $Y \leq (\kappa\ell/2c_2(n))\mathbb{1}_n$. Again, as in equation (4.16) in Theorem 4.5, we split the sum

$$S_\kappa^\Gamma(Z) = \lim_{t \to \infty} \exp\left(-\frac{n\kappa}{4}(\kappa - (n+1))\, t\right) \sum_{\gamma \in \Gamma} K_t^{(\kappa)}(2R^\gamma(Z))$$

in equation (4.8) into sums over $\Gamma \setminus \Gamma_\infty$ and Γ_∞, with $\Gamma_\infty := \Gamma \cap \Gamma_{0,\infty}$, to obtain

$$\begin{aligned}
S_\kappa^\Gamma(Z) \leq\; &\exp\left(-\frac{n\kappa}{4}(\kappa - (n+1))\, t\right) \sum_{\gamma \in \Gamma\backslash\Gamma_\infty} K_t^{(\kappa)}(2R^\gamma(Z)). \\
&+ \lim_{t \to \infty} \exp\left(-\frac{n\kappa}{4}(\kappa - (n+1))\, t\right) \sum_{\gamma \in \Gamma_\infty} K_t^{(\kappa)}(2R^\gamma(Z)).
\end{aligned} \qquad (4.33)$$

Now as $\Gamma \setminus \Gamma_\infty \subseteq \Gamma_0 \setminus \Gamma_{0,\infty}$, expanding the first sum to $\Gamma_0 \setminus \Gamma_{0,\infty}$ and using equation (4.17), we obtain

$$\exp\left(-\frac{n\kappa}{4}(\kappa - (n+1))\, t\right) \sum_{\gamma \in \Gamma\backslash\Gamma_\infty} K_t^{(\kappa)}(2R^\gamma(Z)) \leq c_{n,\Gamma_0}\, \kappa^{n(n+1)/2}.$$

Thus, it only remains to estimate the sum

$$\sum_{\gamma \in \Gamma_\infty} \frac{1}{\prod_{j=1}^n \mathrm{ch}^\kappa(r_j^\gamma(Z))}$$

in equation (4.18). Note that here we now have

$$\Gamma_\infty = \bigcup_{j=0}^{n-1} \Gamma_\infty^j,$$

with $\Gamma_\infty^j = \Gamma \cap \Gamma_{0,\infty}^j$, i.e.,

$$\Gamma_\infty^0 = \left\{ \begin{pmatrix} \mathbb{1}_n & \ell S \\ 0 & \mathbb{1}_n \end{pmatrix} \,\middle|\, S \in \mathrm{Sym}_n(\mathbb{Z}) \right\},$$

$$\Gamma_\infty^j = \left\{ \begin{pmatrix} A & AS \\ 0 & A^{-t} \end{pmatrix} \,\middle|\, A = \begin{pmatrix} \mathbb{1}_j & 0 \\ \ell L & \mathbb{1}_{n-j} \end{pmatrix}, S = \begin{pmatrix} 0 & \ell H^t \\ \ell H & \ell S_2 \end{pmatrix} \right\} \qquad (1 \le j \le (n-1)),$$

where $L, H \in \mathbb{Z}^{(n-j)\times j}$ and $S_2 \in \mathrm{Sym}_{n-j}(\mathbb{Z})$.

Then, for $j = 0$, proceeding as in equation (4.22), we have

$$\sum_{\gamma \in \Gamma_\infty^0} \frac{1}{\prod_{j=1}^n \mathrm{ch}^\kappa(r_j^\gamma(Z))} \le \int_{S \in \mathrm{Sym}_n(\mathbb{R})} \frac{[\mathrm{d}S]}{\det(\mathbb{1}_n + (\frac{1}{2} Y^{-1/2} \ell S Y^{-1/2})^2)^{\kappa/2}}.$$

Now, setting $T = \frac{1}{2} Y^{-1/2} \ell S Y^{-1/2}$, we have

$$[\mathrm{d}T] = c_n \ell^{n(n+1)/2} \det(Y)^{-(n+1)/2} [\mathrm{d}S],$$

thereby giving

$$\sum_{\gamma \in \Gamma_\infty^0} \frac{1}{\prod_{j=1}^n \mathrm{ch}^\kappa(r_j^\gamma(Z))} \le c_n \ell^{-n(n+1)/2} \det(Y)^{(n+1)/2} \int_{T \in \mathrm{Sym}_n(\mathbb{R})} \frac{[\mathrm{d}T]}{\det(\mathbb{1}_n + T^2)^{\kappa/2}}.$$

Now, from $\det(Y) \le (\kappa\ell/(2c_2(n)))^n$, it easily follows that we have an uniform estimate

$$\sum_{\gamma \in \Gamma_\infty^0} \frac{1}{\prod_{j=1}^n \mathrm{ch}^\kappa(r_j^\gamma(Z))} \le c_n \kappa^{n(n+1)/4}$$

independent of the ramification index ℓ.

Similarly, for the $j > 0$ case, proceeding as in Theorem 4.5, in place of substitution equations (4.26)–(4.28), we set

$$2\widetilde{L} = P_2^{-1} \ell L P_1,$$
$$2\widetilde{H} = P_2^{-1} \ell H P_1^{-t},$$
$$2\widetilde{S_2} = P_2^{-1} \ell S_2 P_2^{-t} - P_2^{-1}(\ell H P_1^{-t} P^t + P P_1^{-1} \ell H^t) P_2^{-t},$$

which results in

$$I_{n,\kappa}^j(Z) \le c_n \ell^{-(n-j+1)(n-j)/2 - 2j(n-j)} \det(P_2)^{n-j+1} \det(P_2)^{2j} \int_{\widetilde{L}} \int_{\widetilde{H}} \int_T \frac{[\mathrm{d}T] \wedge [\mathrm{d}\widetilde{H}] \wedge [\mathrm{d}\widetilde{L}]}{|\det(Q + iT)|^\kappa}$$

in place of (4.29). Now, with $\det(P_2) \le c_n(\ell\kappa)^{(n-j)/2}$ coming from $Y \le (\kappa\ell/2c_2(n))\mathbb{1}_n$ via equation (4.31), it follows that

$$\sum_{\gamma \in \Gamma_\infty^j} \frac{1}{\prod_{j=1}^n \mathrm{ch}^\kappa(r_j^\gamma(Z))} I_{n,\kappa}^j(Z) \le c_n \, \ell^{-j(n-j)} \kappa^{(n-j)(n-j+1)/4} \le c_n \kappa^{(n-j)(n-j+1)/4}.$$

Thus, combined, we get an uniform estimate

$$\sum_{\gamma \in \Gamma_\infty} \frac{1}{\prod_{j=1}^n \mathrm{ch}^\kappa(r_j^\gamma(Z))} \le c_n \, \kappa^{n(n+1)/4}$$

resulting in the uniform estimate

$$\lim_{t \to \infty} \exp\left(-\frac{n\kappa}{4}(\kappa - (n+1))\, t \right) \sum_{\gamma \in \Gamma_\infty} K_t^{(\kappa)}(2R^\gamma(Z)) \le c_n \, \kappa^{3n(n+1)/4}$$

in the second sum in equation (4.33), thereby proving the theorem. $\qquad \square$

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